Soundness and Completeness of Symmetric Relators

Sergey Goncharov
University of Birmingham, UK

Pouya Partow
University of Birmingham, UK

Abstract

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There exist various notions of (bi)simulation in the literature. It is usually expected from a symmetric simulation to be a bisimulation. It is true for the most traditionally known notion, but such conclusion must be made with care. We describe a scenario, in which we have a proof that a symmetric relation is a simulation, but the proof does not serve as a proof for the relation to be a bisimulation. We take the Aczel-Mendler simulation[1]. Given a partial order over a functor $F\colon \mathbf{C} \to \mathbf{C}$, a relation r on X is a simulation on a coalgebra (X, α) iff there exists a coalgebra (R, σ) called *witness* that laxly commutes in the following diagram:

$$\begin{tikzcd} X \arrow[d, "\alpha"'] & r \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "\sigma"] \arrow[dl, phantom, "\sqsubseteq"] \arrow[dr, phantom, "\sqsupseteq"] & X \arrow[d, "\alpha"] \\ FX & Fr \arrow[l, "Fp_1"] \arrow[r, "Fp_2"'] & FX \end{tikzcd} \tag{1}$$

If (1) commutes fully, then σ is a witness for r to be an *Aczel-Mendler bisimulation.*

Assuming that $F = \mathcal{P}$ is the powerset functor on **Set**, we take $r = \{(1,2),(2,1),(1,3),(3,1)\}$, and $X = \{1,2,3\}$, and $\alpha(x) = X$ for every $x \in X$, and σ is defined as below:

$$\sigma(w) = \begin{cases} r & w \neq (1,3) \\ r \setminus \{(1,2)\} & w = (1,3) \end{cases}$$

In this scenario, σ is a witness for r to be a simulation, but it is not a witness for r to be a bisimulation, since for every $w \in r$ we have $\alpha(p_1(w)) \subseteq \mathcal{P}p_1(\sigma(w)) = X$. Also, for every $w \in r$, $\mathcal{P}p_2(\sigma(w)) \subseteq \alpha(p_2(w)) = X$. But it is not a bisimulation, since $\alpha(p_2(1,3)) = \alpha(3) = X \neq \mathcal{P}p_2(\sigma(1,3)) = X \setminus \{2\}$.

Inspired by Hermida-Jacobs bisimulation[2] we modify the definition by setting $\mathbf{C}$ to be a regular category, and changing the span (FR, Fp_1, Fp_2) in (1) with the span $((Fr)^\dagger, (Fp_1)^\dagger, (Fp_2)^\dagger)$, where $(Fr)^\dagger$ is the image of $\langle Fp_1, Fp_2\rangle$, and $\langle (Fp_1)^\dagger, (Fp_2)^\dagger\rangle\colon (FR)^\dagger \to FX \times FX$ is monic, so $(Fr)^\dagger$ is a relation. By the symmetry of r there exists $s\colon r \to r$ that we call *swap*, where $p_1 \cdot s = p_2$ and $p_2 \cdot s = p_1$. The necessary and sufficient condition for σ to be a witness for r to be a bisimulation is that

$$\sigma \cdot s = (Fs)^\dagger \cdot \sigma \tag{2}$$

, i.e., s is a F-coalgebra morphism from σ to itself. Nevertheless, still we have the following example of the $\sigma\colon r \to (Fr)^\dagger$ that is a witness for r to be a simulation over (X, α), but it is not a witness for r to be a bisimulation. Keeping the setting of the previous example, we modify σ as follows:

$$\sigma(w) = \begin{cases} (X, X) & w \neq (1,3) \\ (X, X \setminus \{2\}) & w = (1,3) \end{cases}$$

σ is a witness for r to be a simulation since for every $w \in r$ we have $\alpha(p_1(w)) \subseteq ((\mathcal{P}p_1)^\dagger(\sigma(w))) = X$. Also, for every $w \in r$, $((\mathcal{P}p_2)^\dagger(\sigma(w))) \subseteq \alpha(p_2(w)) = X$. But it is not a bisimulation, since $\alpha(p_2(1,3)) = \alpha(3) = X \neq (\mathcal{P}p_2)^\dagger(\sigma(1,3)) = X \setminus \{2\}$. It worth mentioning that σ does not satisfy (2). But still we do not know if having such σ can give us a witness that satisfies (2) or guarantees the existence of such witness.

There is a solution for this problem, if we change the context. If we take our category to be **Set**, we can talk about a kind of well-studied maps named *relator*. For a set functor F, a relator $\mathbf{R}$ takes a relation $r \subseteq X \times Y$ to another relation $\mathbf{R}r \subseteq FX \times FY$. A relator is monotone with respect to inclusion. For example, in our alternative for Aczel-Mendler simulation, for every set functor F, the map that takes a relation r to $(Fr)^\dagger$ is a relator. We show the reverse of a relation r with r^{op}, and composition of r with another relation t with $t \cdot r$. A relation $r \subseteq X \times Y$ is called $\mathbf{R}$-simulation from a coalgebra (X, α) to a coalgebra (Y, β) iff $r \subseteq \beta \cdot \mathbf{R}r \cdot \alpha^{\mathsf{op}}$. Additionally, a relator is called symmetric iff $\mathbf{R}(r^{\mathsf{op}}) = (\mathbf{R}r)^{\mathsf{op}}$, and a $\mathbf{R}$-simulation for a symmetric $\mathbf{R}$ is called a $\mathbf{R}$-bisimulation that resembles (2). $\mathbf{R}$-similarity and $\mathbf{R}$-bisimilarity are the greatest $\mathbf{R}$-simulation and $\mathbf{R}$-bisimulation, respectively. We call a relator $\mathbf{R}$ sound iff $\mathbf{R}$-similarity from a coalgebra (X, α) to a coalgebra (Y, β) is included in the behavioural equivalence from (X, α) to (Y, β), and we call it complete iff the behavioural equivalence is included in the $\mathbf{R}$-similarity.

References

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2 Claudio Hermida and Bart Jacobs. Structural Induction and Coinduction in a Fibrational Setting. *Information and Computation*, 145(2):107–152, 1998. doi:10.1006/inco.1998.2725.