Coalgebraic Simulation.

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Abstract. Hello simulation!

1 Coalgebraic Simulation

We show the category of preorders with monotone functions between them with **PreOrd**. In the diagrams, any arrow that shows a functor, but does not have a label is showing a forgetful functor. Also, we use **Rel** to refer to the category of binary relations. Assuming $R \in \mathbf{Obj}(\mathbf{Rel})$ and $R \subseteq X_1 \times X_2$, and $S \in \mathbf{Obj}(\mathbf{Rel})$ and $S \subseteq Y_1 \times Y_2$, then a morphism $f\colon R \to S$ in this category is the pair (f_1, f_2) of morhpisms in **Set**, where, $f_1\colon X_1 \to Y_1$ and $f_2\colon X_2 \to Y_2$, and for each $(x_1, x_2) \in R$ we have $(f_1(x_1), f_2(x_2)) \in S$. Also, we show projections of $R \in \mathbf{Obj}(\mathbf{Rel})$ with p_1 and p_2 that are morphisms in **Set**.

Definition 1.1 (A Preorder Over a Functor). Assuming $F\colon \mathbf{Set} \to \mathbf{Set}$ is a functor, we call $\sqsubseteq\colon \mathbf{Set} \to \mathbf{PreOrd}$ an order over the functor F iff the following diagram commutes:

$$\begin{tikzcd} & \mathbf{PreOrd} \arrow[d] \\ \mathbf{Set} \arrow[ur, "\sqsubseteq"] \arrow[r, "F"'] & \mathbf{Set} \end{tikzcd}$$

Definition 1.2 (Relation Lifting). Assuming $F\colon \mathbf{Set} \to \mathbf{Set}$ is a functor, then we call $\mathbf{Rel}(F)\colon \mathbf{Rel} \to \mathbf{Rel}$ a relation lifting of F, where the following diagram commutes:

$$\begin{tikzcd} \mathbf{Rel} \arrow[r, "\mathbf{Rel}(F)"] \arrow[d] & \mathbf{Rel} \arrow[d] \\ \mathbf{Set} \times \mathbf{Set} \arrow[r, "F \times F"'] & \mathbf{Set} \times \mathbf{Set} \end{tikzcd}$$

We take $\mathbf{Rel}(F)\colon \mathbf{Rel} \to \mathbf{Rel}$ to be the functor that for an arbitrary functor F takes a relation R, where $R \in \mathbf{Obj}(\mathbf{Rel})$ and $R \subseteq X_1 \times X_2$, and gives the relation that is the image of the function $\langle Fp_1, Fp_2\rangle\colon FR \to FX \times FY$.

Definition 1.3 (Bisimulation). For a functor $F\colon \mathbf{Set} \to \mathbf{Set}$, a bisimulation is a $\mathbf{Rel}(F)$-coalgebra in **Rel**.

Proposition 1.4. *Assuming that (R, α) is a $\mathbf{Rel}(F)$-coalgebra, where $\alpha = \beta_1 \times \beta_2$ in $\mathbf{Set} \times \mathbf{Set}$, then the following diagram commutes, and vice-versa:*

$$\begin{tikzcd}
X_1 \arrow[d, "\beta_1"'] & R \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "\beta"] & X_2 \arrow[d, "\beta_2"] \\
FX_1 & FR \arrow[l, "Fp_1"] \arrow[r, "Fp_2"'] & FX_2
\end{tikzcd}$$

We gave an introduction to Hughes and Jacobs paper. They also have a way to represent simulation relations. In the following, we try to find a suitable formalization for simulation relations, inspired by Hughes and Jacobs.

1.1 Relations as ~~Pullbacks~~ Spans(?)

We can not show every relation by pullbacks, but we can just show relations of the form

$$\{(a, b) \mid f(a) = g(b)\}$$

for some functions f and g, when we are in $\mathbf{Set}$, so we can not show every object in $\mathbf{Rel}$ using this approach, including $\mathbf{Rel}_{\sqsubseteq}(F)(R) \sqsubseteq_{X_2}; \mathbf{Rel}(F)(R); \sqsubseteq_{X_1}$ that is the target of simulation. Although we can show $\mathbf{Rel}_{\sqsubseteq}(F)(R) \sqsubseteq_{X_2}; \mathbf{Rel}(F)(R); \sqsubseteq_{X_1}$ as a span.

Assuming, we have a category $\mathbf{C}$, an object of the category of spans over $\mathbf{C}$ is (R, X_1, X_2, p_1, p_2) in the form of the following diagram:

$$\begin{tikzcd}
& R \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
X_1 & & X_2
\end{tikzcd}$$

A morhpism from a span (R, X_1, X_2, p_1, p_2) to a span (S, Y_1, Y_2, q_1, q_2) is a morphism $f \colon R \to S$ in $\mathbf{C}$, for which exist $f_1 \colon X_1 \to Y_i$ and $f_2 \colon X_2 \to Y_j$, where $i, j \in \{1, 2\}$ and $i \neq j$, and they are in $\mathbf{C}$, that take part in the following commuting diagram:

$$\begin{tikzcd}
X_2 \arrow[d, "f_2"'] & R \arrow[l, "p_2"'] \arrow[r, "p_1"] \arrow[d, "f"] & X_1 \arrow[d, "f_1"] \\
Y_j & S \arrow[l, "q_j"] \arrow[r, "q_i"'] & Y_i
\end{tikzcd}$$

We define a F-simulation as the coalgebra of the object $\sqsubseteq_{X_1};\mathbf{Rel}(F)(R);\sqsubseteq_{X_2}$ that has the following structure in $\mathbf{C}$:

$$\begin{tikzcd}
\sqsubseteq_{X_1};\mathbf{Rel}(F)(R);\sqsubseteq_{X_2} \arrow[r, "\pi_1"] \arrow[dd, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & \sqsubseteq_{X_1};\mathbf{Rel}(F)(R) \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"] \arrow[dr, phantom, "\lrcorner", very near start] & \sqsubseteq_{X_1} \arrow[r, "i_1^1"] \arrow[d, "i_2^1"] & FX_1 \\
 & FR \arrow[r, "Fp_1"] \arrow[d, "Fp_2"'] & FX_1 & \\
\sqsubseteq_{X_2} \arrow[r, "i_1^2"'] \arrow[d, "i_2^2"'] & FX_2 & & \\
FX_2 & & &
\end{tikzcd}$$

We show that if we consider a relation R and its opposite are both simulation relations, then R is a bisimulation. To reach to that goal, we give a formal definition of what we mean by the opposite of R in our categorical setting that we show with R^{op}. $(R^{\mathsf{op}}, p'_1, p'_2)$ is a span, that is isomorphic to R via morphism $s\colon R \to R^{\mathsf{op}}$ in $\mathbf{Rel}$ that we call swap, and it commutes in the following commutative diagram:

$$\begin{tikzcd}
X_2 \arrow[d, "\mathsf{id}"'] & R \arrow[l, "p_2"'] \arrow[r, "p_1"] \arrow[d, "s"'] & X_1 \arrow[d, "\mathsf{id}"] \\
X_2 & R^{\mathsf{op}} \arrow[l, "p'_1"] \arrow[r, "p'_2"'] & X_1
\end{tikzcd}$$

Lemma 1.5. *The relation* $(\sqsubseteq_{X_1};\mathbf{Rel}(F)(R);\sqsubseteq_{X_2})^{\mathsf{op}}$ *is isomorphic to* $\sqsubseteq_{X_2^{\mathsf{op}}}$ $;\mathbf{Rel}(F)(R^{\mathsf{op}});\sqsubseteq_{X_1^{\mathsf{op}}}$.

Proof. We set $s_1\colon \sqsubseteq_{X_1}\to\sqsubseteq^{\mathsf{op}}_{X_1}$ and $s_2\colon \sqsubseteq_{X_2}\to\sqsubseteq^{\mathsf{op}}_{X_2}$ to be the swaps of $\sqsubseteq_{X_1}$ and $\sqsubseteq_{X_2}$, respectively. Since we have

$$\begin{aligned}
& i'^1_1 \cdot s_1 \cdot \varphi_1 \\
&\qquad = i^1_2 \cdot \varphi_2 \\
&\qquad = Fp_1 \cdot \varphi_2 \\
&\qquad = Fp'_2 \cdot Fs \cdot \varphi_2,
\end{aligned}$$

there exists the morphism $s''\colon \sqsubseteq_{X_1};\mathbf{Rel}(F)(R) \to \mathbf{Rel}(F)(R^{\mathsf{op}});\sqsubseteq^{\mathsf{op}}_{X_1}$ depicted in the following commutative diagram:

$$\begin{tikzcd}
\sqsubseteq_{X_1};\mathbf{Rel}(F)(R) \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] \arrow[dr, dashed, "s''"] & \sqsubseteq_{X_1} \arrow[dr, "s_1"] & \\
FR \arrow[dr, "Fs"'] & \mathbf{Rel}(F)(R^{\mathsf{op}});\sqsubseteq^{\mathsf{op}}_{X_1} \arrow[r, "\varphi'_2"] \arrow[d, "\varphi'_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \sqsubseteq^{\mathsf{op}}_{X_1} \arrow[d, "i'^1_1"'] \\
 & FR^{\mathsf{op}} \arrow[r, "Fp'_2"] & FX_1
\end{tikzcd}$$

Similarily, we get $s''^{\text{-}1}\colon\ \sqsubseteq_{X_1}; \mathbf{Rel}(F)(R) \to \mathbf{Rel}(F)(R^{\mathsf{op}}); \sqsubseteq_{X_1}^{\mathsf{op}}$ since

$$
\begin{aligned}
&i^1_2 \cdot s_1^{\text{-}1} \cdot \varphi'_2\\
&\qquad = i'^1_1 \cdot \varphi'_2\\
&\qquad = Fp'_2 \cdot \varphi'_1\\
&\qquad = Fp_1 \cdot Fs_1^{\text{-}1} \cdot \varphi'_1,
\end{aligned}
$$

and it is depicted in the following diagram:

$$
\begin{tikzcd}
\mathbf{Rel}(F)(R^{\mathsf{op}}); \sqsubseteq_{X_1}^{\mathsf{op}} \arrow[r, "\varphi'_2"] \arrow[d, "\varphi'_1"'] \arrow[dr, dashed, "s''^{\text{-}1}"'] & \sqsubseteq_{X_1} \arrow[dr, "s_1^{\text{-}1}"] & \\
FR^{\mathsf{op}} \arrow[dr, "Fs^{\text{-}1}"'] & \sqsubseteq_{X_1}; \mathbf{Rel}(F)(R) \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] \arrow[dr, phantom, "\lrcorner" very near start] & \sqsubseteq_{X_1} \arrow[d, "i^1_2"] \\
& FR \arrow[r, "Fp_1"'] & FX_1
\end{tikzcd}
$$

Obviously, s'' and $s''^{\text{-}1}$ are each other's inverse, thus $\sqsubseteq_{X_1}; \mathbf{Rel}(F)(R)$ and $\mathbf{Rel}(F)(R^{\mathsf{op}}); \sqsubseteq_{X_1}^{\mathsf{op}}$ are isomorphic.

$$
\begin{aligned}
&Fp'_1 \cdot \varphi'_1 \cdot s'' \cdot \pi_1\\
&\qquad = Fp'_1 \cdot Fs \cdot \varphi_2 \cdot \pi_1\\
&\qquad = Fp_2 \cdot \varphi_2 \cdot \pi_1\\
&\qquad = i^2_1 \cdot \pi_2\\
&\qquad = i'^2_2 \cdot s_2 \cdot \pi_2
\end{aligned}
$$

$$
\begin{tikzcd}
\sqsubseteq_{X_1}; \mathbf{Rel}(F)(R); \sqsubseteq_{X_2} \arrow[r, "\pi_1"] \arrow[dd, "\pi_2"'] \arrow[dr, dashed, "s'"] & \sqsubseteq_{X_1}; \mathbf{Rel}(F)(R) \arrow[dr, "s''"] & \\
& \sqsubseteq_{X_2}^{\mathsf{op}}; \mathbf{Rel}(F)(R^{\mathsf{op}}); \sqsubseteq_{X_1}^{\mathsf{op}} \arrow[r, "\pi'_2"] \arrow[dd, "\pi'_1"'] \arrow[ddr, phantom, "\lrcorner" very near start] & \mathbf{Rel}(F)(R^{\mathsf{op}}); \sqsubseteq_{X_1}^{\mathsf{op}} \arrow[d, "\varphi'_1"] \\
\sqsubseteq_{X_2} \arrow[dr, "s_2"] & & FR^{\mathsf{op}} \arrow[d, "Fp'_1"] \\
& \sqsubseteq_{X_2}^{\mathsf{op}} \arrow[r, "i'^2_2"] & FX_2
\end{tikzcd}
$$

$$
\begin{aligned}
&Fp_2 \cdot \varphi_2 \cdot s''^{\text{-}1} \cdot \pi'_2\\
&\qquad = Fp_2 \cdot Fs^{\text{-}1} \cdot \varphi'_1 \cdot \pi'_2\\
&\qquad = Fp'_1 \cdot \varphi'_1 \cdot \pi'_2\\
&\qquad = i'^2_2 \cdot \pi'_1\\
&\qquad = i^2_1 \cdot s_2^{\text{-}1} \cdot \pi'_1
\end{aligned}
$$

$$\begin{tikzcd}
\sqsubseteq^{\mathsf{op}}_{X_2};\mathbf{Rel}(F)(R^{\mathsf{op}});\sqsubseteq^{\mathsf{op}}_{X_1} \arrow[rr, "\pi'_2"] \arrow[dddd, "\pi'_1"'] \arrow[dr, dashed, "s'^{-1}"'] & & \mathbf{Rel}(F)(R^{\mathsf{op}});\sqsubseteq^{\mathsf{op}}_{X_1} \arrow[d, "s''^{-1}"] \\
& \sqsubseteq_{X_1};\mathbf{Rel}(F)(R);\sqsubseteq_{X_2} \arrow[r, "\pi_1"] \arrow[ddd, "\pi_2"'] \arrow[dr, phantom, "\lrcorner"] & \sqsubseteq_{X_1};\mathbf{Rel}(F)(R) \arrow[d, "\varphi_2"] \\
& & FR \arrow[dd, "Fp_2"] \\
& & \\
\sqsubseteq^{\mathsf{op}}_{X_2} \arrow[r, "{s_2}^{-1}"'] & \sqsubseteq_{X_2} \arrow[r, "i^2_1"'] & FX_2
\end{tikzcd}$$

So, we could prove that $\sqsubseteq_{X_1};\mathbf{Rel}(F)(R);\sqsubseteq_{X_2}$ and $\sqsubseteq^{\mathsf{op}}_{X_2};\mathbf{Rel}(F)(R^{\mathsf{op}});\sqsubseteq^{\mathsf{op}}_{X_1}$ are isomorphic. $(\sqsubseteq_{X_1};\mathbf{Rel}(F)(R);\sqsubseteq_{X_2})^{\mathsf{op}}$ is isomorphic to $\sqsubseteq_{X_1};\mathbf{Rel}(F)(R);\sqsubseteq_{X_2}$ by definition, so it is also isomorphic with $\sqsubseteq^{\mathsf{op}}_{X_2};\mathbf{Rel}(F)(R^{\mathsf{op}});\sqsubseteq^{\mathsf{op}}_{X_1}$. $\square$

Proposition 1.6. *Having $\sigma\colon R \to \sqsubseteq_{X_2};\mathbf{Rel}(F)(R);\sqsubseteq_{X_1}$ and $\sigma^{\mathsf{op}}\colon R^{\mathsf{op}} \to \sqsubseteq_{X_1};\mathbf{Rel}(F)(R^{\mathsf{op}});\sqsubseteq_{X_2}$ gives rise to a morphism $\gamma\colon R \to \mathbf{Rel}(F)(R)$, and vice-versa.*

Proof.

$$\begin{tikzcd}
R \arrow[rrrr, "p_1"] \arrow[dr, "\sigma"] \arrow[dddd, "p_2"'] & & & & X_1 \arrow[d, "\alpha"] & \\
& \sqsubseteq_{X_1};\mathbf{Rel}(F)(R);\sqsubseteq_{X_2} \arrow[r, "\pi_1"] \arrow[dd, "\pi_2"'] \arrow[dr, phantom, "\lrcorner"] & \sqsubseteq_{X_1};\mathbf{Rel}(F)(R) \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"] \arrow[dr, phantom, "\lrcorner"] & \sqsubseteq_{X_1} \arrow[r, "i^1_1"] \arrow[d, "i^1_2"] \arrow[dr, "s_1"] & FX_1 & X_1 \arrow[l, "\alpha"'] \\
& & FR \arrow[r, "Fp_1"] \arrow[d, "Fp_2"'] \arrow[dr, "Fs"'] & FX_1 & \sqsubseteq^{\mathsf{op}}_{X_1} \arrow[l, "i'^1_1"'] \arrow[u, "i'^1_2"] & \\
& \sqsubseteq_{X_2} \arrow[r, "i^2_1"'] \arrow[d, "i^2_2"'] \arrow[dr, "s_2"'] & FX_2 & FR^{\mathsf{op}} \arrow[l, "Fp'_1"] \arrow[u, "Fp'_2"] & \mathbf{Rel}(F)(R^{\mathsf{op}});\sqsubseteq^{\mathsf{op}}_{X_1} \arrow[l, "\varphi'_1"'] \arrow[u, "\varphi'_2"'] & \\
X_2 \arrow[r, "\beta"'] & FX_2 & \sqsubseteq^{\mathsf{op}}_{X_2} \arrow[l, "i'^2_1"] \arrow[u, "i'^2_2"] & & \sqsubseteq^{\mathsf{op}}_{X_2};\mathbf{Rel}(F)(R^{\mathsf{op}});\sqsubseteq^{\mathsf{op}}_{X_1} \arrow[ll, "\pi'_1"] \arrow[u, "\pi'_2"'] & \\
& X_2 \arrow[u, "\beta"'] & & & & R^{\mathsf{op}} \arrow[llll, "p'_1"'] \arrow[ul, "\sigma^{\mathsf{op}}"'] \arrow[uuuu, "p'_2"']
\end{tikzcd}$$

($\Leftarrow$): We assume that we have the morphism $\gamma\colon R \to FR$ such that the following diagram commutes:

$$\begin{tikzcd}
X_2 \arrow[d, "\beta"'] & R \arrow[l, "p_2"'] \arrow[r, "p_1"] \arrow[d, "\gamma"] & X_1 \arrow[d, "\alpha"] \\
FX_2 & FR \arrow[l, "Fp_2"] \arrow[r, "Fp_1"'] & FX_1
\end{tikzcd} \tag{1}$$

Since $\sqsubseteq_{X_1}$ and $\sqsubseteq_{X_1}$ preorders, they each have a morphism refl that pre-composed with their projections gives identity. As it is depicted in the following diagram the pullback property of $\sqsubseteq_{X_1};\mathbf{Rel}(F)(R)$ gives us $\sigma'\colon R \to \sqsubseteq_{X_1};\mathbf{Rel}(F)(R)$ in the following commutative diagram:

$$\begin{tikzcd}
R \arrow[r, "p_1"] \arrow[dr, dashed, "\sigma'"] \arrow[ddr, bend right, "\gamma"'] & X_1 \arrow[r, "\alpha"] & FX_1 \arrow[d, "\mathsf{refl}"] \\
 & \sqsubseteq_{X_1}; \mathbf{Rel}(F)(R) \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"] \arrow[dr, phantom, "\lrcorner"] & \sqsubseteq_{X_1} \arrow[d, "i_2^1"] \\
 & FR \arrow[r, "Fp_1"'] & FX_1
\end{tikzcd} \tag{2}$$

{eq:diag-thm-sig'}

Then the pullback property of $\sqsubseteq_{X_1}; \mathbf{Rel}(F)(R); \sqsubseteq_{X_2}$ gives us the existence of $\sigma \colon R \to \sqsubseteq_{X_1}; \mathbf{Rel}(F)(R); \sqsubseteq_{X_2}$ in the following commutative diagram:

{eq:diag-thm-sig}

$$\begin{tikzcd}
X_2 \arrow[ddd, "\beta"'] & & & R \arrow[lll, "p_2"'] \arrow[dll, dashed, "\sigma"'] \arrow[dl, "\sigma'"] \arrow[ddl, bend left, "\gamma"] \\
 & \sqsubseteq_{X_1}; \mathbf{Rel}(F)(R); \sqsubseteq_{X_2} \arrow[r, "\pi_1"] \arrow[dd, "\pi_2"'] \arrow[dr, phantom, "\lrcorner"] & \sqsubseteq_{X_1}; \mathbf{Rel}(F)(R) \arrow[d, "\varphi_2"] & \\
 & & FR \arrow[d, "Fp_2"] & \\
FX_2 \arrow[r, "\mathsf{refl}"'] & \sqsubseteq_{X_2} \arrow[r, "i_1^2"'] & FX_2 &
\end{tikzcd} \tag{3}$$

Now, we show that σ is a simulation:

$$\begin{aligned}
& i_1^1 \cdot \varphi_1 \cdot \pi_1 \cdot \sigma \\
& \qquad = i_1^1 \cdot \varphi_1 \cdot \sigma' && /\!/ \ (3) \\
& \qquad = i_1^1 \cdot \mathsf{refl} \cdot \alpha \cdot p_1 && /\!/ \ (2) \\
& \qquad = \alpha \cdot p_1
\end{aligned}$$

$$\begin{aligned}
& i_2^2 \cdot \pi_2 \cdot \sigma \\
& \qquad = i_2^2 \cdot \mathsf{refl} \cdot \beta \cdot p_2 && /\!/ \ (3) \\
& \qquad = \beta \cdot p_2
\end{aligned}$$

Considering that $s \colon R \to R^{\mathsf{op}}$ and $s' \colon \sqsubseteq_{X_1}; \mathbf{Rel}(F)(R); \sqsubseteq_{X_2} \to \sqsubseteq_{X_2}^{\mathsf{op}}; \mathbf{Rel}(F)(R^{\mathsf{op}}); \sqsubseteq_{X_1}^{\mathsf{op}}$ are swapping isomorphisms, We set $\sigma^{\mathsf{op}} \colon R^{\mathsf{op}} \to \sqsubseteq_{X_2}^{\mathsf{op}}; \mathbf{Rel}(F)(R^{\mathsf{op}}); \sqsubseteq_{X_1}^{\mathsf{op}}$ to be $\sigma^{\mathsf{op}} = s' \cdot \sigma \cdot s^{\text{-}1}$. Now, we show that σ' is a simulation:

$$\begin{aligned}
& i_2'^1 \cdot \varphi_2' \cdot \pi_2' \cdot \sigma^{\mathsf{op}} \\
& \qquad = i_2'^1 \cdot \varphi_2' \cdot \pi_2' \cdot s' \cdot \sigma \cdot s^{\text{-}1} \\
& \qquad = i_1^1 \cdot \varphi_1 \cdot \pi_1 \cdot \sigma \cdot s^{\text{-}1} \\
& \qquad = \alpha \cdot p_1 \cdot s^{\text{-}1}
\end{aligned}$$

$$
\begin{aligned}
&= \alpha \cdot p'_2
\end{aligned}
$$

$$
\begin{aligned}
i'^2_1 \cdot \pi'_1 \cdot \sigma^{\mathsf{op}} = & \\
&= i'^2_1 \cdot \pi'_1 \cdot s' \cdot \sigma \cdot s^{-1} \\
&= i^2_2 \cdot \pi_2 \cdot \sigma \cdot s^{-1} \\
&= \beta \cdot p_2 \cdot s^{-1} \\
&= \beta \cdot p'_1
\end{aligned}
$$

1.2 Simulation with one relation composition

We recall everything we had in the previous section. Although we want to work with the functor that takes $R \subseteq X_1 \times X_2$ and gives $\mathbf{Rel}(F)(R);\sqsubseteq_{X_2}$.

$$
\begin{tikzcd}
\mathbf{Rel}(F)(R);\sqsubseteq_{X_2} \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & FR \arrow[r, "Fp_1"] \arrow[d, "Fp_2"] & FX_1 \\
\sqsubseteq_{X_2} \arrow[r, "i_1"'] \arrow[d, "i_2"'] & FX_2 & \\
FX_2 & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
R \arrow[rrr, "p_1"] \arrow[dr, "\sigma"] \arrow[ddd, "p_2"'] & & & X_1 \arrow[d, "\alpha"] & \\
& \mathbf{Rel}(F)(R);\sqsubseteq_{X_2} \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & FR \arrow[r, "Fp_1"] \arrow[d, "Fp_2"] & FX_1 & X_1 \arrow[l, "\alpha"'] \\
& \sqsubseteq_{X_2} \arrow[r, "i_1"'] \arrow[d, "i_2"'] & FX_2 & FR \arrow[l, "Fp_2"] \arrow[u, "Fp_1"'] & \\
X_2 \arrow[r, "\beta"'] & FX_2 & \sqsubseteq^{\mathsf{op}}_{X_2} \arrow[l, "i'_2"] \arrow[u, "i'_1"] & \mathbf{Rel}(F)(R);\sqsubseteq^{\mathsf{op}}_{X_2} \arrow[l, "\pi'_2"] \arrow[u, "\pi'_1"'] & \\
& X_2 \arrow[u, "\beta"] & & & R \arrow[lll, "p_2"] \arrow[ul, "\sigma^{\mathsf{op}}"'] \arrow[uuu]
\end{tikzcd}
$$

Proposition 1.7. *Assuming $R \subseteq X \times X$, then if we have $\sigma\colon R \to \mathbf{Rel}(F)(R);\sqsubseteq_X$ as a simulation for R, and R is reflexive, then we have $\gamma\colon R \to \mathbf{Rel}(F)(R)$ as a bisimulation for R, and vice-versa.*

Proof. $(\Rightarrow)$:

$$
\begin{aligned}
Fp_2 \cdot \pi_1 \cdot \sigma & \\
&= i_1 \cdot \pi_2 \cdot \sigma \\
&= i'_2 \cdot s \cdot \pi_2 \cdot \sigma \\
&=
\end{aligned}
$$

1.3 Using Lax Pullbacks (Comma Objects) to Model Simulation

A big concern with this approach is that Comma Objects are defined in a 2-category, so we can not define them in **Set**, while our main inspirational example is coming from **Set**.

1.4 Working in Set First, Like Hughes and Jacobs

1.5 Choosing a suitable order for our setting

Maybe we can first choose a suitable order on $T(\Sigma_{\mathsf{v}}\mu\Sigma \times D(\mu\Sigma, \mu\Sigma))$ and then prove that if a relation and its inverse is a simulation then it is a bisimulation as well. Maybe T being ω-continuous can give the ordering. It can be something easier that relates to termination as well! That if a term has a big-step evaluation, then it is bigger than or equal to any other term, and if it does not, then it is less than or equal to any other term.

2 Symmetric Simulation is Bisimulation

Definition 2.1 (Graph). In a category $\mathbf{C}$ a graph is a tuple (R, X) of the following form:

$$\begin{tikzcd} & R \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & X \end{tikzcd}$$

Graphs over $\mathbf{C}$ form a category that we show by $\mathbf{Gra}(\mathbf{C})$.

Definition 2.2 (Symmetric Graph). A graph (R, X) is symmetric iff there exists an endomorphism $s\colon R \to R$, such that the following diagram commutes

$$\begin{tikzcd} X \arrow[d, "\mathsf{id}"'] & R \arrow[l, "p_2"'] \arrow[r, "p_1"] \arrow[d, "s"] & X \arrow[d, "\mathsf{id}"] \\ X & R \arrow[l, "p_1"] \arrow[r, "p_2"'] & X \end{tikzcd}$$

and $s \cdot s = \mathsf{id}$. We call s a *swap* for R.

{lem:gra-sym}

Lemma 2.3. *Symmetry of a graphs over preserved a functor.*

Definition 2.4 (Relation). A relation in a category $\mathbf{C}$ is a graph (R, X) where $\langle p_1, p_2\rangle\colon R \to X \times Y$ is monic. Relations over $\mathbf{C}$ form a category that we show by $\mathbf{Rel}(\mathbf{C})$.

Definition 2.5 (Jointly Monic). A pair of morphisms $p_1, p_2\colon R \to X$ is jointly monic iff for every pair of morphisms $f, g\colon A \to R$ assuming that $p_1 \cdot f = p_1 \cdot g$ and $p_2 \cdot f = p_2 \cdot g$ then $f = g$.

{prop:rel-joi-mon}

Proposition 2.6. *A graph (R, X) is a relation iff p_1 and p_2 are jointly monic.*

Proof. ($\Rightarrow$): We assume that for morphisms $f, g\colon A \to R$ we have $p_1 \cdot f = p_1 \cdot g$ and $p_2 \cdot f = p_2 \cdot g$, and we want to prove that $f = g$. Assuming that $\pi_1, \pi_2\colon X \times X \to X$ are projections of $X \times X$, then we have:

$$\begin{aligned}
&\langle p_1, p_2\rangle \cdot f \\
&\qquad = \langle p_1 \cdot f, p_2 \cdot f\rangle \\
&\qquad = \langle p_1 \cdot g, p_2 \cdot g\rangle \\
&\qquad = \langle p_1, p_2\rangle \cdot g
\end{aligned}$$

Since $\langle p_1, p_2\rangle$ is moinc, from $\langle p_1, p_2\rangle \cdot f = \langle p_1, p_2\rangle \cdot g$ we get $f = g$.

($\Leftarrow$): Assuming for some morphisms $f, g\colon A \to R$ we have $\langle p_1, p_2\rangle \cdot f = \langle p_1, p_2\rangle \cdot g$ we need to prove $f = g$. From $\langle p_1, p_2\rangle \cdot f = \langle p_1, p_2\rangle \cdot g$ we get $\langle p_1 \cdot f, p_2 \cdot f\rangle = \langle p_1 \cdot g, p_2 \cdot g\rangle$. Assuming that $\pi_1, \pi_2\colon X \times X \to X$ are projections of $X \times X$, then we have $\pi_1 \cdot \langle p_1 \cdot f, p_2 \cdot f\rangle = \pi_1 \cdot \langle p_1 \cdot g, p_2 \cdot g\rangle$, and then $p_1 \cdot f = p_1 \cdot g$. Similarly we also get $p_2 \cdot f = p_2 \cdot g$. So, since p_1 and p_2 are jointly monic, then we have $f = g$. $\square$

We need to work with endofunctors over $\mathbf{C}$ that are lifted over $\mathbf{Rel}(\mathbf{C})$, for which we need to first define endofunctors lifted over $\mathbf{Gra}(\mathbf{C})$. Lifting from $\mathbf{C}$ to $\mathbf{Gra}(\mathbf{C})$ is easy. For $F\colon \mathbf{C} \to \mathbf{C}$ we define $F_{\mathbf{Gra}}\colon \mathbf{Gra}(\mathbf{C}) \to \mathbf{Gra}(\mathbf{C})$ as a functor that takes a graph (R, X), and gives (FR, FX), were F is also applied on legs of the graph, i.e., $p_1, p_2\colon R \to X$, so, we get the following graph:

$$\begin{tikzcd}
& FR \arrow[dl, "Fp_1"'] \arrow[dr, "Fp_2"] & \\
FX & & FX
\end{tikzcd}$$

This lifting does not work for $\mathbf{Rel}$. As an example, if we set F to be the powerset functor $\mathcal{P}$, then $(\mathcal{P}R, \mathcal{P}X)$ is not necessarily a relation anymore. For example, if we take $R = \{(1,0), (0,1), (0,0), (1,1)\}$, then taking $\{\{(1,0), (0,1), (0,0), (1,1)\}\}$ and $\{\{(1,0), (0,1), (0,0)\}\}$ as elements of $\mathcal{P}R$, the morphism $\langle \mathcal{P}p_1, \mathcal{P}p_2\rangle$ maps them both to $(\{0,1\}, \{0,1\})$ so it is not monic.

To cope with this, we assume the following epi-mono decomposition for $(R, X) \in \mathbf{Rel}(\mathbf{C})$:

$$\begin{tikzcd}
R \arrow[r, two heads, "e_R"'] \arrow[rr, bend left, "\langle p_1, p_2\rangle"] & R^{\dagger} \arrow[r, tail, "\langle p_1^{\dagger}, p_2^{\dagger}\rangle"'] & X \times X
\end{tikzcd}$$

We can define $(-)^\dagger$ as a functor from $\mathbf{Gra}(\mathbf{C}) \to \mathbf{Rel}(\mathbf{C})$, then we define $F_{\mathbf{Rel}}\colon \mathbf{Rel}(\mathbf{C}) \to \mathbf{Rel}(\mathbf{C})$ to take (R, X) to the following relation:

$$
\begin{tikzcd}
 & (FR)^\dagger \arrow[dl, "(Fp_1)^\dagger"'] \arrow[dr, "(Fp_2)^\dagger"] \arrow[dd, dashed, tail, "\langle (Fp_1)^\dagger,(Fp_2)^\dagger\rangle"] & \\
FX & & FX \\
 & FX \times FX &
\end{tikzcd}
$$

{lem:norm-simp}

Lemma 2.7. *Assuming that we have the following commutative diagram:*

$$
\begin{tikzcd}
X \arrow[d, "\alpha"'] & R \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "\sigma"] & X \arrow[d, "\alpha"] \\
FX & FR \arrow[l, "Fp_1"] \arrow[r, "Fp_2"'] & FX
\end{tikzcd}
$$

Then there exists $\sigma^\dagger\colon R \to (FR)^\dagger$ in the following diagram that is also commutative:

$$
\begin{tikzcd}
X \arrow[d, "\alpha"'] & R \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "\sigma^\dagger"] & X \arrow[d, "\alpha"] \\
FX & (FR)^\dagger \arrow[l, "Fp_1^\dagger"] \arrow[r, "Fp_2^\dagger"'] & FX
\end{tikzcd}
$$

Proof. The proof is trivial considering that $\sigma^\dagger = e_{FR} \cdot \sigma$, where e_{FR} is the epimorphism in the epi-mono factorization of $\langle Fp_1, Fp_2\rangle$, as depicted in the following diagram:

$$
\begin{tikzcd}
X \arrow[d, "\alpha"'] & R \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "\sigma"] & X \arrow[d, "\alpha"] \\
FX \arrow[d, "\mathsf{id}"'] & FR \arrow[l, "Fp_1"] \arrow[r, "Fp_2"'] \arrow[d, "e_{FR}"] & FX \arrow[d, "\mathsf{id}"] \\
FX & (FR)^\dagger \arrow[l, "Fp_1^\dagger"] \arrow[r, "Fp_2^\dagger"'] & FX
\end{tikzcd}
$$

□

We show this relation with $F_{\mathbf{Rel}}(R, X)$.

{def:sim}

Definition 2.8 (Simulation). A coalgebra $\sigma\colon R \to (FR)^\dagger$ is a simulation over the F-coalgebra $\alpha\colon X \to FX$ iff the following diagram is lax-commutative:

{eq:diag-lax-sim}

$$
\begin{tikzcd}
X \arrow[d, "\alpha"'] \arrow[dr, phantom, "\sqsubseteq"] & R \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "\sigma"] \arrow[dr, phantom, "\sqsupseteq"] & X \arrow[d, "\alpha"] \\
FX & (FR)^\dagger \arrow[l, "(Fp_1)^\dagger"] \arrow[r, "(Fp_2)^\dagger"'] & FX
\end{tikzcd}
\tag{4}
$$

{def:bisim} **Definition 2.9 (Bisimulation).** The morphism σ in Definition 2.8 is a bisimulation iff the mentioned diagram is fully commutative.

Remark 2.10. The mentioned definition of bisimulation is actually, the classical one in the literature that is to have the following commutative diagram:

$$\begin{tikzcd} X \arrow[d, "\alpha"'] & R \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "\sigma"] & X \arrow[d, "\alpha"] \\ FX & (FR)^\dagger \arrow[l, "(Fp_1)^\dagger"] \arrow[r, "(Fp_2)^\dagger"'] & FX \end{tikzcd}$$

It may look different because we have FX and not $(FX)^\dagger$, but they are the same. An object $X \in \mathbf{Obj}(\mathbf{C})$ is $(X, X) \in \mathbf{Rel}(\mathbf{C})$ having id as its legs. Meaning that the $(FX)^\dagger = FX$.

{prop:iff-sim-bsim}

Proposition 2.11. *Assuming that we have a bisimulation σ for R, we have the following equation:*

$$\sigma \cdot s = (Fs)^\dagger \cdot \sigma$$

Proof. We recall that by Lemma 2.3, $F_{\mathbf{Rel}}(R, X)$ is symmetric with the swap $(Fs)^\dagger$. Assuming that σ is a bisimulation, we have the following commutative diagram:

$$\begin{tikzcd} & R \arrow[dl, bend right, "p_2"'] \arrow[dr, bend left, "p_1"] \arrow[d, "s"] & \\ X \arrow[d, "\alpha"'] & R \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "\sigma"] & X \arrow[d, "\alpha"] \\ FX & (FR)^\dagger \arrow[l, "(Fp_1)^\dagger"] \arrow[r, "(Fp_2)^\dagger"'] \arrow[d, "(Fs)^\dagger"] & FX \\ & (FR)^\dagger \arrow[ul, bend left, "(Fp_2)^\dagger"] \arrow[ur, bend right, "(Fp_1)^\dagger"'] & \end{tikzcd} \tag{5}$$ {eq:diag-sym-rel}

And it entails that the following diagrams are also commutative:

$$\begin{tikzcd} X \arrow[d, "\alpha"'] & R \arrow[l, "p_2"'] \arrow[r, "p_1"] \arrow[d, "\sigma \cdot s"] & X \arrow[d, "\alpha"] \\ FX & (FR)^\dagger \arrow[l, "(Fp_1)^\dagger"] \arrow[r, "(Fp_2)^\dagger"'] & FX \end{tikzcd} \qquad \begin{tikzcd} X \arrow[d, "\alpha"'] & R \arrow[l, "p_2"'] \arrow[r, "p_1"] \arrow[d, "(Fs)^\dagger \cdot \sigma"] & X \arrow[d, "\alpha"] \\ FX & (FR)^\dagger \arrow[l, "(Fp_1)^\dagger"] \arrow[r, "(Fp_2)^\dagger"'] & FX \end{tikzcd}$$

So, since $(Fp_1)^\dagger$ and $(Fp_2)^\dagger$ are jointly monic (because $F_{\mathbf{Rel}}(R, X)$ is a relation and Proposition 2.6) we have $\sigma \cdot s = (Fs)^\dagger \cdot \sigma$. □

Corollary 2.12. *Assuming σ_1 and σ_2 are simulations of type $R \to (FR)^\dagger$, and R is symmetric and both σ_1 and σ_2 satisfy the following property:*

$$(Fs)^\dagger \cdot \sigma = \sigma \cdot s$$

Then $\sigma_1 = \sigma_2$.

Proof. As the mentioned property is equivalent with σ being a bisimulation, and bisimulation is unique, then $\sigma_1 = \sigma_2$.

Now, we give a counter example of a symmetric relation on **Set** that is a simulation according to Definition 2.8, i.e, exists the morphism σ that commutes laxly in (4), but σ is not a coalgebraic bisimulation, although the relation that we give is clearly a bisimulation in the classic sense. We set $R = \{(A, B), (B, A), (C_1, C_2), (C_2, C_1), (C_2', C_2), (C_2, C_2'), (C_2, C_2)\}$, $F = \mathbf{Id}$, $\sqsubseteq = \Delta \cup \{(C_1, C_2), (C_2, C_2')\}$, and the coalgebra α is defined with the following set of reductions:

$$A \to C_1 \qquad B \to C_2 \qquad C_1 \to C_1 \qquad C_2 \to C_2 \qquad C_2' \to C_2$$

And finally, we define σ as follows:

$$\sigma(w) = \begin{cases} (\alpha \cdot p_1(w), \alpha \cdot p_2(w)) & w \neq (B, A) \\ (C_2', C_2) & w = (B, A) \end{cases}$$

It is easy to check that the conditions $\alpha \cdot p_1 \sqsubseteq (Fp_1)^\dagger \cdot \sigma$ and $(Fp_2)^\dagger \cdot \sigma \sqsubseteq \alpha \cdot p_2$ are satisfied. For every $w \in R$ if $w \neq (B, A)$ then for $i \in \{1, 2\}$, we have $\alpha \cdot p_i = (Fp_i)^\dagger \cdot \sigma$, and for $w = (B, A)$ we have $\alpha \cdot p_1(B, A) = C_2 \sqsubseteq C_2' = (Fp_1)^\dagger \cdot \sigma(B, A)$, and $\alpha \cdot p_2(B, A) = C_1 \sqsubseteq C_2 = (Fp_2)^\dagger \cdot \sigma(B, A)$. And σ is not a coalgebraic bisimulation as $\alpha \cdot p_1(B, A) = C_2 \neq C_2' = (Fp_1)^\dagger \cdot \sigma(B, A)$.

An interesting question would be to find out what conditions σ should have (maybe we have the answer to this! Proposition 2.11), or how it should be constructed (perhaps based on a given poset) so that it will also be a coalgebraic bisimulation if R is symmetric. Another avenue would be to give another definition for simulation that does not have this issue.

Well! This counter example does not work! Because the described order $\sqsubseteq$ does not satisfy the condition mentioned in Jacobs's paper. The condition is that the order on FX should satisfy the property that for a morphism $f \colon X \to Y$ the morphism $Ff \colon FX \to FY$ preserves $\sqsubseteq$. Probably, the only poset that has this property for **Id** is Δ. If there is a counter-example, it is true for another functor.

(But still!)We have a counter-example for a symmetric relation R that has a witness to be a simulation, but that morphism does not serve as a witness for R to be a bisimulation. In the category of sets we assume that $F = \mathcal{P}$, and take $R = \{(1, 2), (2, 1), (1, 3), (3, 1)\}$, and $X = \{1, 2, 3\}$. $\alpha(x) = X$ for every $x \in X$, and σ is defined as below:

$$\sigma(w) = \begin{cases} (X, X) & w \neq (1, 3) \\ (X, X \backslash \{2\}) & w = (1, 3) \end{cases}$$

In this scenario, σ is a witness for R to be a simulation, but it is not a witness for R to be a bisimulation. σ is a witness for R to be a simulation since for every $w \in R$ we have $\alpha(p_1(w)) \subseteq ((\mathcal{P}p_1)^\ddagger(\sigma(w))) = X$. Also, for every $w \in R$, $((\mathcal{P}p_2)^\ddagger(\sigma(w))) \subseteq \alpha(p_2(w)) = X$. But it is not a bisimulation, since $\alpha(p_2(1, 3)) = \alpha(3) = X \neq (\mathcal{P}p_2)^\ddagger(\sigma(1, 3)) = X \backslash \{2\}$.

Example 2.13. And another counter-example!!! Assume that $F = \mathcal{P}$, and take $R = X \times X \backslash \{(1,3),(3,1)\}$, and $X = \{1,2,3\}$. α is defined as below:

$$\alpha(x) = \begin{cases} \{1,2\} & x = 1 \\ \{2,3\} & x = 2 \\ \{3\} & x = 3 \end{cases}$$

And σ_1 is defined as below:

$$\sigma_1(w) = \begin{cases} \{(1,2),(2,2)\} & w = (1,2) \\ \{(2,1),(3,2)\} & w = (2,1) \\ \{(1,2),(2,1)\} & w = (1,1) \\ \{(2,2),(3,3)\} & w \in \{(2,2),(3,2)\} \\ \{(2,3),(3,3)\} & w = (2,3) \\ \{(3,3)\} & w = (3,3) \end{cases}$$

$$(\mathcal{P}p_1)^\dagger \cdot \sigma_1(w) = \begin{cases} \{1,2\} & w = (1,2) \\ \{2,3\} & w = (2,1) \\ \{1,2\} & w = (1,1) \\ \{2,3\} & w \in \{(2,2),(3,2)\} \\ \{2,3\} & w = (2,3) \\ \{3\} & w = (3,3) \end{cases} (\mathcal{P}p_2)^\dagger \cdot \sigma_1(w) = \begin{cases} \{2\} & w = (1,2) \\ \{1,2\} & w = (2,1) \\ \{1,2\} & w = (1,1) \\ \{2,3\} & w \in \{(2,2),(3,2)\} \\ \{3\} & w = (2,3) \\ \{3\} & w = (3,3) \end{cases} \blacksquare$$

$$\sigma_1'(w) = \begin{cases} \{(1,2),(2,2),(2,3)\} & w = (1,2) \\ \{(2,1),(2,2),(3,2)\} & w = (2,1) \\ \{(1,2),(2,1)\} & w = (1,1) \\ \{(2,2),(3,3)\} & w = (2,2) \\ \{(2,2),(3,3),(3,2)\} & w = (3,2) \\ \{(2,3),(2,2),(3,3)\} & w = (2,3) \\ \{(3,3)\} & w = (3,3) \end{cases}$$

$$(\mathcal{P}p_1)^\dagger \cdot \sigma_1'(w) = \begin{cases} \{1,2\} & w = (1,2) \\ \{2,3\} & w = (2,1) \\ \{1,2\} & w = (1,1) \\ \{2,3\} & w = (2,2) \\ \{2,3\} & w = (3,2) \\ \{2,3\} & w = (2,3) \\ \{3\} & w = (3,3) \end{cases} (\mathcal{P}p_2)^\dagger \cdot \sigma_1'(w) = \begin{cases} \{2,3\} & w = (1,2) \\ \{1,2\} & w = (2,1) \\ \{1,2\} & w = (1,1) \\ \{2,3\} & w = (2,2) \\ \{2,3\} & w = (3,2) \\ \{2,3\} & w = (2,3) \\ \{3\} & w = (3,3) \end{cases}$$

σ_1' is not a simulation!

$$\sigma_1''(w) = \begin{cases} \{(1,2)\} & w = (1,2) \\ \{(2,1)\} & w = (2,1) \\ \{(1,2),(2,1)\} & w = (1,1) \\ \{(2,2),(3,3)\} & w = (2,2) \\ \{(3,3)\} & w = (3,2) \\ \{(3,3)\} & w = (2,3) \\ \{(3,3)\} & w = (3,3) \end{cases}$$

$$\begin{aligned} \sigma_1 &\sqsubseteq \sigma_1' \\ \sigma_3 &\sqsubseteq \sigma_1' \\ \beta &= \sigma_1' \end{aligned}$$

$$(\mathcal{P}s)^\dagger \cdot \sigma_1 \cdot s(w) = \begin{cases} \{(1,2),(2,3)\} & w = (1,2) \\ \{(2,1),(2,2)\} & w = (2,1) \\ \{(1,2),(2,1)\} & w = (1,1) \\ \{(2,2),(3,3)\} & w = (2,2) \\ \{(3,2),(3,3)\} & w = (3,2) \\ \{(2,2),(3,3)\} & w = (2,3) \\ \{(3,3)\} & w = (3,3) \end{cases}$$

$$(\mathcal{P}p_1)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma_1 \cdot s(w) = \begin{cases} \{1,2\} & w = (1,2) \\ \{2\} & w = (2,1) \\ \{1,2\} & w = (1,1) \\ \{2,3\} & w = (2,2) \\ \{3\} & w = (3,2) \\ \{2,3\} & w = (2,3) \\ \{3\} & w = (3,3) \end{cases} (\mathcal{P}p_2)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma_1 \cdot s(w) = \begin{cases} \{2,3\} & w = (1,2) \\ \{1,2\} & w = (2,1) \\ \{1,2\} & w = (1,1) \\ \{2,3\} & w = (2,2) \\ \{2,3\} & w = (3,2) \\ \{2,3\} & w = (2,3) \\ \{3\} & w = (3,3) \end{cases}$$

In this scenario, σ_1 is a simulation, but it is not a bisimulation. σ_1', σ_1'' and $(\mathcal{P}s)^\dagger \cdot \sigma_1 \cdot s$ are neither. We can not make σ_1 bigger here to make it a bisimulation as $\alpha \cdot p_1(3,2) = \{3\} \subsetneq \{2,3\} = (\mathcal{P}p_1)^\dagger \cdot \sigma_1(3,2)$.

The following is also a simulation and not a bisimulation:

$$\sigma_2(w) = \begin{cases} \{(1,2),(2,2)\} & w = (1,2) \\ \{(2,1),(3,2)\} & w = (2,1) \\ \{(1,2),(2,1)\} & w = (1,1) \\ \{(2,2),(3,3)\} & w = (2,2) \\ \{(2,3),(3,3)\} & w = (2,3) \\ \{(3,3)\} & w \in \{(3,2),(3,3)\} \end{cases}$$

$$(\mathcal{P}p_1)^\dagger \cdot \sigma_2(w) = \begin{cases} \{1,2\} & w = (1,2) \\ \{2,3\} & w = (2,1) \\ \{1,2\} & w = (1,1) \\ \{2,3\} & w = (2,2) \\ \{2,3\} & w = (2,3) \\ \{3\} & w \in \{(3,2),(3,3)\} \end{cases} \qquad (\mathcal{P}p_2)^\dagger \cdot \sigma_2(w) = \begin{cases} \{2\} & w = (1,2) \\ \{1,2\} & w = (2,1) \\ \{1,2\} & w = (1,1) \\ \{2,3\} & w = (2,2) \\ \{3\} & w = (2,3) \\ \{3\} & w \in \{(3,2),(3,3) \end{cases}$$

$$\sigma_2'(w) = \begin{cases} \{(1,2),(2,2),(2,3)\} & w = (1,2) \\ \{(2,1),(2,2),(3,2)\} & w = (2,1) \\ \{(1,2),(2,1)\} & w = (1,1) \\ \{(2,2),(3,3)\} & w = (2,2) \\ \{(3,3),(3,2)\} & w = (3,2) \\ \{(2,3),(3,3)\} & w = (2,3) \\ \{(3,3)\} & w = (3,3) \end{cases}$$

$$(\mathcal{P}p_1)^\dagger \cdot \sigma_2'(w) = \begin{cases} \{1,2\} & w = (1,2) \\ \{2,3\} & w = (2,1) \\ \{1,2\} & w = (1,1) \\ \{2,3\} & w = (2,2) \\ \{3\} & w = (3,2) \\ \{2,3\} & w = (2,3) \\ \{3\} & w = (3,3) \end{cases} (\mathcal{P}p_2)^\dagger \cdot \sigma_2'(w) = \begin{cases} \{2,3\} & w = (1,2) \\ \{1,2\} & w = (2,1) \\ \{1,2\} & w = (1,1) \\ \{2,3\} & w = (2,2) \\ \{2,3\} & w = (3,2) \\ \{3\} & w = (2,3) \\ \{3\} & w = (3,3) \end{cases}$$

$$(\mathcal{P}s)^\dagger \cdot \sigma_2 \cdot s(w) = \begin{cases} \{(1,2),(2,3)\} & w = (1,2) \\ \{(2,1),(2,2)\} & w = (2,1) \\ \{(1,2),(2,1)\} & w = (1,1) \\ \{(2,2),(3,3)\} & w = (2,2) \\ \{(3,3),(3,2)\} & w = (3,2) \\ \{(3,3)\} & w = (2,3) \\ \{(3,3)\} & w = (3,3) \end{cases}$$

$$(\mathcal{P}p_1)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma_2 \cdot s(w) = \begin{cases} \{1,2\} & w = (1,2) \\ \{2\} & w = (2,1) \\ \{1,2\} & w = (1,1) \\ \{2,3\} & w = (2,2) \\ \{3\} & w = (3,2) \\ \{2\} & w = (2,3) \\ \{3\} & w = (3,3) \end{cases} (\mathcal{P}p_2)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma_2 \cdot s(w) = \begin{cases} \{2,3\} & w = (1,2) \\ \{1,2\} & w = (2,1) \\ \{1,2\} & w = (1,1) \\ \{2,3\} & w = (2,2) \\ \{2,3\} & w = (3,2) \\ \{3\} & w = (2,3) \\ \{3\} & w = (3,3) \end{cases}$$

σ_2 is a simulation, σ_2' is a bisimulation, and $(\mathcal{P}s)^\dagger \cdot \sigma_2 \cdot s$ is neither. The following is both a simulation and a bisimulation:

$$\sigma_3(w) = \begin{cases} \{(1,2),(2,2),(2,3)\} & w=(1,2) \\ \{(2,1),(2,2),(3,2)\} & w=(2,1) \\ \{(1,2),(2,1)\} & w=(1,1) \\ \{(2,2),(3,3)\} & w=(2,2) \\ \{(2,3),(3,3)\} & w=(2,3) \\ \{(3,2),(3,3)\} & w=(3,2) \\ \{(3,3)\} & w=(3,3) \end{cases}$$

$$(\mathcal{P}p_1)^\dagger \cdot \sigma_3(w) = \begin{cases} \{1,2\} & w=(1,2) \\ \{2,3\} & w=(2,1) \\ \{1,2\} & w=(1,1) \\ \{2,3\} & w=(2,2) \\ \{2,3\} & w=(2,3) \\ \{3\} & w=(3,2) \\ \{3\} & w=(3,3) \end{cases} (\mathcal{P}p_2)^\dagger \cdot \sigma_3(w) = \begin{cases} \{2,3\} & w=(1,2) \\ \{1,2\} & w=(2,1) \\ \{1,2\} & w=(1,1) \\ \{2,3\} & w=(2,2) \\ \{3\} & w=(2,3) \\ \{2,3\} & w=(3,2) \\ \{3\} & w=(3,3) \end{cases}$$

The following is also a simulation and not a bisimulation:

$$\sigma_4(w) = \begin{cases} \{(1,2),(2,2),(3,2),(2,3),(3,3)\} & w=(1,2) \\ \{(1,1),(2,1),(1,2),(2,2),(3,2)\} & w=(2,1) \\ \{(1,2),(2,1)\} & w=(1,1) \\ \{(2,2),(3,3)\} & w=(2,2) \\ \{(2,3),(3,3)\} & w=(2,3) \\ \{(1,2),(2,2),(3,2),(2,3),(3,3)\} & w=(3,2) \\ \{(3,3)\} & w=(3,3) \end{cases}$$

$$(\mathcal{P}p_1)^\dagger \cdot \sigma_4(w) = \begin{cases} \{1,2,3\} & w=(1,2) \\ \{1,2,3\} & w=(2,1) \\ \{1,2\} & w=(1,1) \\ \{2,3\} & w=(2,2) \\ \{2,3\} & w=(2,3) \\ \{1,2,3\} & w=(3,2) \\ \{3\} & w=(3,3) \end{cases} (\mathcal{P}p_2)^\dagger \cdot \sigma_4(w) = \begin{cases} \{2,3\} & w=(1,2) \\ \{1,2\} & w=(2,1) \\ \{1,2\} & w=(1,1) \\ \{2,3\} & w=(2,2) \\ \{3\} & w=(2,3) \\ \{2,3\} & w=(3,2) \\ \{3\} & w=(3,3) \end{cases}$$

$$\sigma_4''(w) = \begin{cases} \{(1,2),(2,2),(2,3)\} & w = (1,2) \\ \{(2,1),(2,2),(3,2)\} & w = (2,1) \\ \{(1,2),(2,1)\} & w = (1,1) \\ \{(2,2),(3,3)\} & w = (2,2) \\ \{(2,3),(3,3)\} & w = (2,3) \\ \{(3,2),(3,3)\} & w = (3,2) \\ \{(3,3)\} & w = (3,3) \end{cases}$$

$$(\mathcal{P}p_1)^\dagger \cdot \sigma_4''(w) = \begin{cases} \{1,2\} & w = (1,2) \\ \{2,3\} & w = (2,1) \\ \{1,2\} & w = (1,1) \\ \{2,3\} & w = (2,2) \\ \{2,3\} & w = (2,3) \\ \{3\} & w = (3,2) \\ \{3\} & w = (3,3) \end{cases} \quad (\mathcal{P}p_2)^\dagger \cdot \sigma_4''(w) = \begin{cases} \{2,3\} & w = (1,2) \\ \{1,2\} & w = (2,1) \\ \{1,2\} & w = (1,1) \\ \{2,3\} & w = (2,2) \\ \{3\} & w = (2,3) \\ \{2,3\} & w = (3,2) \\ \{3\} & w = (3,3) \end{cases}$$

$$(\mathcal{P}s)^\dagger \cdot \sigma_4 \cdot s(w) = \begin{cases} \{(1,1),(1,2),(2,1),(2,2),(2,3)\} & w = (1,2) \\ \{(2,1),(2,2),(2,3),(3,2),(3,3)\} & w = (2,1) \\ \{(1,2),(2,1)\} & w = (1,1) \\ \{(2,2),(3,3)\} & w = (2,2) \\ \{(2,1),(2,2),(2,3),(3,2),(3,3)\} & w = (2,3) \\ \{(3,2),(3,3)\} & w = (3,2) \\ \{(3,3)\} & w = (3,3) \end{cases}$$

$$(\mathcal{P}p_1)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma_4 \cdot s(w) = \begin{cases} \{1,2\} & w = (1,2) \\ \{2,3\} & w = (2,1) \\ \{1,2\} & w = (1,1) \\ \{2,3\} & w = (2,2) \\ \{2,3\} & w = (2,3) \\ \{3\} & w = (3,2) \\ \{3\} & w = (3,3) \end{cases} \quad (\mathcal{P}p_2)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma_4 \cdot s(w) = \begin{cases} \{1,2,3\} & w = (1,2) \\ \{1,2,3\} & w = (2,1) \\ \{1,2\} & w = (1,1) \\ \{2,3\} & w = (2,2) \\ \{1,2,3\} & w = (2,3) \\ \{2,3\} & w = (3,2) \\ \{3\} & w = (3,3) \end{cases} \blacksquare$$

σ_4 is a simulation, σ_4'' is a bisimulation, and $(\mathcal{P}s)^\dagger \cdot \sigma_4 \cdot s$ is neither.

The following is also a simulation and not a bisimulation:

$$\sigma_5(w) = \begin{cases} \{(1,2),(2,2)\} & w=(1,2) \\ \{(2,2),(3,2)\} & w=(2,1) \\ \{(1,1),(2,1)\} & w=(1,1) \\ \{(2,2),(3,2)\} & w=(2,2) \\ \{(2,3),(3,3)\} & w=(2,3) \\ \{(3,3)\} & w=(3,2) \\ \{(3,3)\} & w=(3,3) \end{cases}$$

$$(\mathcal{P}p_1)^\dagger \cdot \sigma_5(w) = \begin{cases} \{1,2\} & w=(1,2) \\ \{2,3\} & w=(2,1) \\ \{1,2\} & w=(1,1) \\ \{2,3\} & w=(2,2) \\ \{2,3\} & w=(2,3) \\ \{3\} & w=(3,2) \\ \{3\} & w=(3,3) \end{cases} (\mathcal{P}p_2)^\dagger \cdot \sigma_5(w) = \begin{cases} \{2\} & w=(1,2) \\ \{2\} & w=(2,1) \\ \{1\} & w=(1,1) \\ \{2\} & w=(2,2) \\ \{3\} & w=(2,3) \\ \{3\} & w=(3,2) \\ \{3\} & w=(3,3) \end{cases}$$

The following is also a simulation and not a bisimulation:

$$\sigma_6(w) = \begin{cases} \{(1,2),(2,2)\} & w=(1,2) \\ \{(2,1),(3,1)\} & w=(2,1) \\ \{(1,2),(2,1)\} & w=(1,1) \\ \{(2,3),(3,3)\} & w=(2,2) \\ \{(2,3),(3,3)\} & w=(2,3) \\ \{(3,2)\} & w=(3,2) \\ \{(3,3)\} & w=(3,3) \end{cases}$$

$$(\mathcal{P}p_1)^\dagger \cdot \sigma_6(w) = \begin{cases} \{1,2\} & w=(1,2) \\ \{2,3\} & w=(2,1) \\ \{1,2\} & w=(1,1) \\ \{2,3\} & w=(2,2) \\ \{2,3\} & w=(2,3) \\ \{3\} & w=(3,2) \\ \{3\} & w=(3,3) \end{cases} (\mathcal{P}p_2)^\dagger \cdot \sigma_6(w) = \begin{cases} \{2\} & w=(1,2) \\ \{1\} & w=(2,1) \\ \{2\} & w=(1,1) \\ \{3\} & w=(2,2) \\ \{3\} & w=(2,3) \\ \{2\} & w=(3,2) \\ \{3\} & w=(3,3) \end{cases}$$

$$\sigma_2' = \sigma_5' = \sigma_6' = \sigma_3 = \sigma_4''$$

If we define $\sqsubseteq$ on simulations as

$$\sigma_1 \sqsubseteq \sigma_2 \iff \forall x_1, x_2 \in X, (\mathcal{P}p_i)^\dagger \cdot \sigma_1(x_1,x_2) \subseteq (\mathcal{P}p_i)^\dagger \cdot \sigma_2(x_1,x_2)$$

Lemma 2.14. $(Hom(R, (\mathcal{P}R)^\dagger), \sqsubseteq)$ *is a poset.*

Proof. Reflexivity and transitivity are obvious. We need to prove anti-symmetry.
TODO: Finish!

Then we have

$$\begin{array}{ccccccc} \sigma_6 & & & & \sigma_1 & & \\ & \rotatebox{45}{$\sqsubseteq$} & & \rotatebox{-45}{$\sqsubseteq$} & & \rotatebox{45}{$\sqsubseteq$} & \\ \sigma_5 & \sqsubseteq & \sigma_2 & \sqsubseteq & \sigma_3 & \sqsubseteq & \sigma_4 \end{array}$$

We recall that in the above diagram σ_3 is a bisimulation, and the rest are simulations.

{def:join-meet}

Definition 2.15. We define $\sqcup$ and $\sqcap$ on morphisms as follows:

$$\forall x_1, x_2 \in X,$$
$$\sigma_1 \sqcup \sigma_2(x_1, x_2) = \sigma_1(x_1, x_2) \cup \sigma_2(x_1, x_2),$$
$$\sigma_1 \sqcap \sigma_2(x_1, x_2) = (\mathcal{P}p_1)^\dagger \cdot \sigma_1(x_1, x_2) \cap (\mathcal{P}p_1)^\dagger \cdot \sigma_2(x_1, x_2) \times (\mathcal{P}p_2)^\dagger \cdot \sigma_1(x_1, x_2) \cap (\mathcal{P}p_2)^\dagger \cdot \sigma_2(x_1, x_2).$$

{lem:proj-dist-set}

Lemma 2.16. *For relations R_1 and R_2 the following equation holds:*

$$(\mathcal{P}p_i)(R_1 \cup R_2) = (\mathcal{P}p_i)(R_1) \cup (\mathcal{P}p_i)(R_2)$$

Proof. We prove the lemma for the case that $i = 1$. The proof is the same for $i = 2$. Assuming $x_1 \in (\mathcal{P}p_1)^\dagger(R_1 \cup R_2)$ then exists x_2 that $(x_1, x_2) \in R_1 \cup R_2$, thus either $(x_1, x_2) \in R_1$ or $(x_1, x_2) \in R_2$, so we have $x_1 \in (\mathcal{P}p_1)^\dagger(R_1)$ or $x_1 \in (\mathcal{P}p_1)^\dagger(R_2)$, respectively. So, we have $x_1 \in (\mathcal{P}p_1)^\dagger(R_1) \cup (\mathcal{P}p_1)^\dagger(R_2)$.

Now, assuming that $x_1 \in (\mathcal{P}p_1)^\dagger(R_1) \cup (\mathcal{P}p_1)^\dagger(R_2)$ either $x_1 \in (\mathcal{P}p_1)^\dagger(R_1)$ or $x_1 \in (\mathcal{P}p_1)^\dagger(R_2)$. Without loss of generality, we can assume $x_1 \in (\mathcal{P}p_1)^\dagger(R_j)$, where $j \in \{1, 2\}$. Then there exists x_2 that $(x_1, x_2) \in R_j$, then we have $(x_1, x_2) \in R_1 \cup R_2$ that gives $x_1 \in (\mathcal{P}p_1)^\dagger(R_1 \cup R_2)$.

Lemma 2.17. *Assuming that σ_1 and σ_2 are simulation structures of type $R \to (\mathcal{P}R)^\dagger$, then $\sigma_1 \sqcup \sigma_2$ and $\sigma_1 \sqcap \sigma_2$ are also simulation structures of the same type.*

Proof. Since σ_1 and σ_2 are simulation structures, for every $(x_1, x_2) \in R$, for $i \in \{1, 2\}$ we have:

$$\alpha(x_1) \subseteq (\mathcal{P}p_1)^\dagger \cdot \sigma_i(x_1, x_2), \tag{6}$$
$$(\mathcal{P}p_2)^\dagger \cdot \sigma_i(x_1, x_2) \subseteq \alpha(x_2). \tag{7}$$

First, we prove the case for $\sqcup$. Since $\alpha(x_1) \subseteq (\mathcal{P}p_1)^\dagger \cdot \sigma_i(x_1, x_2)$ we have the following:

$$\alpha(x_1) \subseteq (\mathcal{P}p_1)^\dagger \cdot \sigma_1(x_1, x_2) \cup (\mathcal{P}p_1)^\dagger \cdot \sigma_2(x_1, x_2)$$

So, by Lemma 2.16 we have $\alpha(x_1) \subseteq (\mathcal{P}p_1)^\dagger(\sigma_1(x_1, x_2) \cup \sigma_2(x_1, x_2))$. Similarly, we have $(\mathcal{P}p_2)^\dagger \cdot \sigma_i(x_1, x_2) \subseteq \alpha(x_2)$ that gives the following:

$$(\mathcal{P}p_2)^\dagger \cdot \sigma_1(x_1, x_2) \cup (\mathcal{P}p_2)^\dagger \cdot \sigma_2(x_1, x_2) \subseteq \alpha(x_2)$$

So, by Lemma 2.16 we have $(\mathcal{P}p_2)^\dagger(\sigma_1(x_1,x_2) \cup \sigma_2(x_1,x_2)) \subseteq \alpha(x_2)$.

Now, we prove the case for $\sqcap$. For $\sqcap$ unlike $\sqcup$ we need to prove that $\sigma_1 \sqcap \sigma_2(x_1,x_2) \in (\mathcal{P}R)^\dagger$. To achieve this, we need to show that assuming π_1, π_2 are projections of $\sigma_1 \sqcap \sigma_2(x_1,x_2)$, then for $j \in \{1,2\}$ we have $\pi_j \cdot (\sigma_1 \sqcap \sigma_2)(x_1,x_2) \subseteq \mathcal{P}p_j(R)$. Since $(\mathcal{P}p_j)^\dagger \cdot \sigma_i(x_1,x_2) \subseteq \mathcal{P}p_j(R)$, we have $\pi_j \cdot (\sigma_1 \sqcap \sigma_2)(x_1,x_2) \subseteq \mathcal{P}p_j(R)$, so we have $\sigma_1 \sqcap \sigma_2(x_1,x_2) \in (\mathcal{P}R)^\dagger$, meaning that $\pi_j \cdot (\sigma_1 \sqcap \sigma_2)(x_1,x_2) = (\mathcal{P}p_j)^\dagger \cdot (\sigma_1 \sqcap \sigma_2)(x_1,x_2)$.[1]

For $j \in \{1,2\}$ we have

{eq:proj-meet}

$$(\mathcal{P}p_j)^\dagger \cdot (\sigma_1 \sqcap \sigma_2(x_1,x_2)) = (\mathcal{P}p_j)^\dagger \cdot \sigma_1(x_1,x_2) \cap (\mathcal{P}p_j)^\dagger \cdot \sigma_2(x_1,x_2). \quad (8)$$

Since $\alpha(x_1) \subseteq (\mathcal{P}p_1)^\dagger \cdot \sigma_i(x_1,x_2)$, we have

$$\alpha(x_1) \subseteq (\mathcal{P}p_1)^\dagger \cdot \sigma_1(x_1,x_2) \cap (\mathcal{P}p_1)^\dagger \cdot \sigma_2(x_1,x_2),$$

so by (8) we have $\alpha(x_1) \subseteq (\mathcal{P}p_1)^\dagger \cdot (\sigma_1 \sqcap \sigma_2(x_1,x_2))$. Similarly, since $(\mathcal{P}p_2)^\dagger \cdot \sigma_i(x_1,x_2) \subseteq \alpha(x_2)$, we have

$$(\mathcal{P}p_2)^\dagger \cdot \sigma_1(x_1,x_2) \cap (\mathcal{P}p_2)^\dagger \cdot \sigma_2(x_1,x_2) \subseteq \alpha(x_2),$$

so by (8) we have $(\mathcal{P}p_2)^\dagger \cdot (\sigma_1 \sqcap \sigma_2(x_1,x_2)) \subseteq \alpha(x_2)$. □

{lem:sim-opsim-inc}

Lemma 2.18. *Assuming that $\sigma\colon R \to (\mathcal{P}R)^\dagger$ is a simulation structure, and R is symmetric, then for all $(x_1,x_2) \in R$ we have:*

1. $(\mathcal{P}p_1)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma \cdot s(x_1,x_2) \subseteq (\mathcal{P}p_1)^\dagger \cdot \sigma(x_1,x_2)$
2. $(\mathcal{P}p_2)^\dagger \cdot \sigma(x_1,x_2) \subseteq (\mathcal{P}p_2)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma \cdot s(x_1,x_2)$

Proof. We prove the second clause. By (4) for every $(x_1,x_2) \in R$ we have

$$\alpha(x_1) \subseteq (\mathcal{P}p_1)^\dagger \cdot \sigma(x_1,x_2),$$
$$(\mathcal{P}p_2)^\dagger \cdot \sigma(x_1,x_2) \subseteq \alpha(x_2).$$

From $\alpha(x_1) \subseteq (\mathcal{P}p_1)^\dagger \cdot \sigma(x_1,x_2)$ since R is symmetric we get $\alpha(x_2) \subseteq (\mathcal{P}p_1)^\dagger \cdot \sigma(x_2,x_1)$, where

$$(\mathcal{P}p_1)^\dagger \cdot \sigma(x_2,x_1) = (\mathcal{P}p_2)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma \cdot s(x_1,x_2).$$

So, from $(\mathcal{P}p_2)^\dagger \cdot \sigma(x_1,x_2) \subseteq \alpha(x_2)$ we have $(\mathcal{P}p_2)^\dagger \cdot \sigma(x_1,x_2) \subseteq (\mathcal{P}p_2)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma \cdot s(x_1,x_2)$. Similarly, we can get the other inequation. □

{lem:sim-bisim-inc}

Lemma 2.19. *Assuming that $\sigma\colon R \to (\mathcal{P}R)^\dagger$ is a simulation structure, and $\beta\colon R \to (\mathcal{P}R)^\dagger$ is a bisimulation structure,*

---

[1] PP Note: The last part of the proof is necessary because the type of the codomain of the definition of $\sqcap$ is not $(\mathcal{P}R)^\dagger$, but it is $\mathcal{P}X \times \mathcal{P}X$. Perhaps the epi-mono factorization must be used to cope with this in the abstract case.

1. *if $\sigma \sqsubseteq \beta$ then we have:*

$$\alpha(x_1) = (\mathcal{P}p_1)^{\dagger} \cdot \sigma(x_1, x_2),$$

and if R is symmetric we have

$$(\mathcal{P}p_2)^{\dagger} \cdot (\mathcal{P}s)^{\dagger} \cdot \sigma \cdot s(x_1, x_2) = \alpha(x_2).$$

2. *if $\beta \sqsubseteq \sigma$ then we have:*

$$(\mathcal{P}p_2)^{\dagger} \cdot \sigma(x_1, x_2) = \alpha(x_2)$$

and if R is symmetric we have

$$\alpha(x_1) = (\mathcal{P}p_1)^{\dagger} \cdot (\mathcal{P}s)^{\dagger} \cdot \sigma \cdot s(x_1, x_2).$$

Proof. 1. Since σ is a simulation structure for an arbitrary $(x_1, x_2) \in R$ we have $\alpha(x_1) \subseteq (\mathcal{P}p_1)^{\dagger} \cdot \sigma(x_1, x_2)$. Since $\sigma \sqsubseteq \beta$ we have $(\mathcal{P}p_1) \cdot \sigma(x_1, x_2) \subseteq (\mathcal{P}p_1) \cdot \beta(x_1, x_2)$, while $(\mathcal{P}p_1) \cdot \beta(x_1, x_2) = \alpha(x_1)$ by definition of bisimulation. So we have $\alpha(x_1) = (\mathcal{P}p_1)^{\dagger} \cdot \sigma(x_1, x_2)$. Then because of the symmetry of R the second clause is easily achievable by using the equations in (5).

2. This clause can be proven similar to (1).

□

Proposition 2.20. *Assuming that $\sigma \colon R \to (\mathcal{P}R)^{\dagger}$ is a simulation structure, and $\beta \colon R \to (\mathcal{P}R)^{\dagger}$ is a bisimulation structure,*

1. *if $\sigma \sqsubseteq \beta$ then we have:*

$$\beta = \sigma \sqcup ((\mathcal{P}s)^{\dagger} \cdot \sigma \cdot s)$$

2. *if $\beta \sqsubseteq \sigma$ then we have:*

$$\beta = \sigma \sqcap ((\mathcal{P}s)^{\dagger} \cdot \sigma \cdot s)$$

Proof. 1. We need to prove that $\sigma \sqcup ((\mathcal{P}s)^{\dagger} \cdot \sigma \cdot s)$ is the bisimulation structure. By Lemma 2.18.(1), for every $(x_1, x_2) \in R$, we have $(\mathcal{P}p_1)^{\dagger} \cdot (\mathcal{P}s)^{\dagger} \cdot \sigma \cdot s(x_1, x_2) \subseteq (\mathcal{P}p_1)^{\dagger} \cdot \sigma(x_1, x_2)$, and by Lemma 2.19.(1), we have $(\mathcal{P}p_1)^{\dagger} \cdot \sigma(x_1, x_2) = \alpha(x_1)$. So, we have $(\mathcal{P}p_1)^{\dagger} \cdot \sigma \sqcup (\mathcal{P}p_1)^{\dagger} \cdot (\mathcal{P}s)^{\dagger} \cdot \sigma \cdot s(x_1, x_2) = \alpha(x_1)$, then by Lemma 2.16 we have $(\mathcal{P}p_1)^{\dagger} \cdot (\sigma \sqcup ((\mathcal{P}s)^{\dagger} \cdot \sigma \cdot s))(x_1, x_2) = \alpha(x_1)$.

Also, by Lemma 2.19.(1) we have $(\mathcal{P}p_2)^{\dagger} \cdot (\mathcal{P}s)^{\dagger} \cdot \sigma(x_1, x_2) = \alpha(x_2)$. So, since we already have $(\mathcal{P}p_2)^{\dagger} \cdot \sigma(x_1, x_2) \subseteq \alpha(x_2)$ then by Lemma 2.16 we have $(\mathcal{P}p_2)^{\dagger} \cdot (\sigma \sqcup ((\mathcal{P}s)^{\dagger} \cdot \sigma \cdot s))(x_1, x_2) = \alpha(x_2)$.

2. We need to prove that $\sigma \sqcap ((\mathcal{P}s)^{\dagger} \cdot \sigma \cdot s)$ is the bisimulation structure. For $i \in \{1, 2\}$, for every $(x_1, x_2) \in R$, we have:

$$\begin{aligned}
&(\mathcal{P}p_i)^{\dagger} \cdot (\sigma \sqcap ((\mathcal{P}s)^{\dagger} \cdot \sigma \cdot s))(x_1, x_2) \\
&= (\mathcal{P}p_i)^{\dagger} \cdot (((\mathcal{P}p_1)^{\dagger} \cdot \sigma(x_1, x_2) \cap (\mathcal{P}p_1)^{\dagger} \cdot ((\mathcal{P}s)^{\dagger} \cdot \sigma \cdot s)(x_1, x_2)) \times ((\mathcal{P}p_2)^{\dagger} \cdot \sigma(x_1, x_2) \cap (\mathcal{P}p_2)^{\dagger} \cdot ((\mathcal{P}s)^{\dagger} \cdot \sigma \cdot s)(x_1, x_2))) \\
&= (\mathcal{P}p_i)^{\dagger} \cdot \sigma(x_1, x_2) \cap (\mathcal{P}p_i)^{\dagger} \cdot ((\mathcal{P}s)^{\dagger} \cdot \sigma \cdot s)(x_1, x_2)
\end{aligned}$$

By Lemma 2.18.(1), $(\mathcal{P}p_1)^\dagger \cdot ((\mathcal{P}s)^\dagger \cdot \sigma \cdot s)(x_1, x_2) \subseteq (\mathcal{P}p_1)^\dagger \cdot \sigma(x_1, x_2)$, and by Lemma 2.19.(2) we have $(\mathcal{P}p_1)^\dagger \cdot ((\mathcal{P}s)^\dagger \cdot \sigma \cdot s)(x_1, x_2) = \alpha(x_1)$, so we have $(\mathcal{P}p_1)^\dagger \cdot (\sigma \sqcap ((\mathcal{P}s)^\dagger \cdot \sigma \cdot s))(x_1, x_2) = \alpha(x_1)$.

Also, by Lemma 2.19.(2) we have $(\mathcal{P}p_2)^\dagger \cdot \sigma(x_1, x_2) = \alpha(x_2)$, so, since by Lemma 2.18.(2), we have $(\mathcal{P}p_2)^\dagger \cdot \sigma(x_1, x_2) \subseteq (\mathcal{P}p_2)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma \cdot s(x_1, x_2)$, so we have $(\mathcal{P}p_2)^\dagger \cdot (\sigma \sqcap ((\mathcal{P}s)^\dagger \cdot \sigma \cdot s)(x_1, x_2)) = \alpha(x_2)$. □

Corollary 2.21. *Assuming that R is a symmetric relation, and $S \neq \emptyset$ is the set of all simulation structures of the type $R \to (\mathcal{P}R)^\dagger$, then if the bisimulation morphism exists, it is equal with the following morphism:*

$$(\bigsqcup_{\sigma\in S} \sigma) \sqcap (\mathcal{P}s)^\dagger \cdot (\bigsqcup_{\sigma\in S} \sigma) \cdot s$$

Lemma 2.22. *For every $S \in \mathcal{P}R$,*

$$((\mathcal{P}p_1)(S), (\mathcal{P}p_2)(S)) \in (\mathcal{P}R)^\dagger \Leftrightarrow (\mathcal{P}p_1)(S) \subseteq (\mathcal{P}p_1)(R), (\mathcal{P}p_2)(S) \subseteq (\mathcal{P}p_2)(R)$$

{lem:alph-prod}

Lemma 2.23. *Assuming that R is a symmetric relation, and $S \neq \emptyset$ is the set of all simulation structures of the type $R \to (\mathcal{P}R)^\dagger$, then there there exists a simulation structure $\sigma \in S$ that for every (x_1, x_2), $(\mathcal{P}p_1)^\dagger \cdot \sigma(x_1, x_2) = \alpha(x_1)$.*

Proof. Since $S \neq \emptyset$ there exists $\delta \in S$. We define σ for every (x_1, x_2) as the following:

$$\sigma(x_1, x_2) = (\alpha(x_1), (\mathcal{P}p_2)^\dagger \cdot \delta(x_1, x_2))$$

We have $\sigma(x_1, x_2) \in (\mathcal{P}R)^\dagger$, as $\alpha(x_1) \subseteq \mathcal{P}p_1(R)$ and $(\mathcal{P}p_2)^\dagger \cdot \delta(x_1, x_2) \subseteq \mathcal{P}p_2(R)$ are inherited from δ being a simulation structure. Also, it obviously is a simulation as $(\mathcal{P}p_1)^\dagger \cdot \sigma(x_1, x_2) = \alpha(x_1)$ and $(\mathcal{P}p_2)^\dagger \cdot \sigma(x_1, x_2) \subseteq \alpha(x_2)$ as $(\mathcal{P}p_2)^\dagger \cdot \delta(x_1, x_2) \subseteq \alpha(x_2)$.

{prop:sym-rel-bisim}

Proposition 2.24. *Assuming that R is a symmetric relation, and $S \neq \emptyset$ is the set of all simulation structures of the type $R \to (\mathcal{P}R)^\dagger$, then the following morphism is the bisimulation structure:*

$$(\bigsqcap_{\sigma\in S} \sigma) \sqcup (\mathcal{P}s)^\dagger \cdot (\bigsqcap_{\sigma\in S} \sigma) \cdot s$$

Proof. For every $(x_1, x_2) \in R$ we have

$$(\mathcal{P}p_1)^\dagger \cdot ((\bigsqcap_{\sigma\in S} \sigma) \sqcup (\mathcal{P}s)^\dagger \cdot (\bigsqcap_{\sigma\in S} \sigma) \cdot s)(x_1, x_2) = (\mathcal{P}p_1)^\dagger \cdot (\bigsqcap_{\sigma\in S} \sigma)(x_1, x_2),$$

and

$$(\mathcal{P}p_2)^\dagger \cdot ((\bigsqcap_{\sigma\in S} \sigma) \sqcup (\mathcal{P}s)^\dagger \cdot (\bigsqcap_{\sigma\in S} \sigma) \cdot s)(x_1, x_2) = (\mathcal{P}p_2)^\dagger \cdot ((\mathcal{P}s)^\dagger \cdot (\bigsqcap_{\sigma\in S} \sigma) \cdot s)(x_1, x_2).$$

By Lemma 2.23 there exists a simulation $\delta \in S$ for which we have $(\mathcal{P}p_1)^\dagger \cdot \delta(x_1, x_2) = \alpha(x_1)$. So, $(\mathcal{P}p_1)^\dagger \cdot (\bigsqcap_{\sigma\in S} \sigma)(x_1, x_2) = \alpha(x_1)$. Then by the equations in (5) we also get $(\mathcal{P}p_2)^\dagger \cdot ((\mathcal{P}s)^\dagger \cdot (\bigsqcap_{\sigma\in S} \sigma) \cdot s)(x_1, x_2) = \alpha(x_2)$. □

3 Symmetric Simulation in Quantaloids

We generalize Proposition 2.24 in regular quantaloids. A quantaloid is a category enriched with suplattices. Abstractly, first we define an operation that we need on morphisms that takes two simulation witnesses of type $R \to (FR)^{\dagger}$ to a morphism of type $R \to FX \times FX$:

$$\sigma_1 \boxtimes \sigma_2 = (Fp_1)^{\dagger} \cdot \sigma_1 \sqcap (Fp_1)^{\dagger} \cdot \sigma_2 \times (Fp_2)^{\dagger} \cdot \sigma_1 \sqcap (Fp_2)^{\dagger} \cdot \sigma_2$$

{lem:alph-prod-abs}

Lemma 3.1. *Assuming that R is a symmetric relation, and $S \neq \emptyset$ is the set of all simulation witnesses of the type $R \to (FR)^{\dagger}$, then there exists a simulation witness $\sigma \in S$ that, $(Fp_1)^{\dagger} \cdot \sigma = \alpha \cdot p_1$.*

Proof. Since $S \neq \emptyset$ there exists $\delta \in S$. We define σ as the following:

$$\sigma = \langle (\alpha \cdot p_1), (Fp_2)^{\dagger} \cdot \delta \rangle$$

We have $(Fp_1)^{\dagger} \cdot \sigma$.

4 Relators

4.1 Two-way similarity in Hughes-Jacobs

Hughes and Jacobs define two-way similarity as $\leqslant \cap \leqslant^{\mathsf{op}}$. They give a sufficient condition for the two-way similarity to be the bisimilarity. We discuss that this condition does not allow us to say that a symmetric simularity is a bisimilarity. The condition is:

$$\sqsubseteq; (FR_1)^{\dagger}; \sqsubseteq \quad \cap \quad \sqsubseteq^{\mathsf{op}}; (FR_2)^{\dagger}; \sqsubseteq^{\mathsf{op}} \qquad \subseteq \qquad (F(R_1 \cap R_2))^{\dagger}$$

We need the case that $R_1 = R_2$, and we refer to it with R. So, we need to have

$$\sqsubseteq; (FR)^{\dagger}; \sqsubseteq \quad \cap \quad \sqsubseteq^{\mathsf{op}}; (FR)^{\dagger}; \sqsubseteq^{\mathsf{op}} \qquad \subseteq \qquad (FR)^{\dagger}.$$

Assuming $(x_1, x_2) \in \sqsubseteq; (FR)^{\dagger}; \sqsubseteq \cap \sqsubseteq^{\mathsf{op}}; (FR)^{\dagger}; \sqsubseteq^{\mathsf{op}}$ means that there exist $(u, v), (u', v') \in (FR)^{\dagger}$, such that $x_1 \sqsubseteq u$, $u' \sqsubseteq x_1$, $v \sqsubseteq x_2$, and $x_2 \sqsubseteq v'$. We can also use the symmetry of R, and then from $(x_1, x_2) \in R$ derive that there exist $(w, z), (w', z')$ such that $x_1 \sqsubseteq z$, $z' \sqsubseteq x_1$, $w \sqsubseteq x_2$, and $x_2 \sqsubseteq w'$. The situation can be illustrated as the following:

$$\begin{array}{ccccccccc}
u' & & & & u & v & & & v' \\
 & \rotatebox{45}{$\sqsubseteq$} & & \rotatebox{-45}{$\sqsupseteq$} & & & \rotatebox{45}{$\sqsubseteq$} & \rotatebox{-45}{$\sqsupseteq$} & \\
 & & x_1 & & & & x_2 & & \\
 & \rotatebox{-45}{$\sqsupseteq$} & & \rotatebox{45}{$\sqsubseteq$} & & & \rotatebox{-45}{$\sqsupseteq$} & \rotatebox{45}{$\sqsubseteq$} & \\
z' & & & & z & w & & & w'
\end{array}$$

But then how can we derive $(x_1, x_2) \in (FR)^{\dagger}$?

TODO: Investigate more! Can it be doable really?!

4.2 Uniqueness of the witness in Hughes-Jacobs definition

In this section we set **Rel** to be the category that has sets as objects and binary relations as morphisms. We answer the question that why there can be multiple simulation witnesses based on Definition 2.8, while for the same relation, there is only one witness according to Hughes-Jacobs simulation.

Definition 4.1 (Hughes-Jacobs Simulation). For a functor F, and a poset $\sqsubseteq$ over F a HuJ-simulation is a relation r for which there exists a morphism $\sigma\colon r \to (Fr)^\dagger$ called *witness* such that the following diagram commutes (; is the relation composition):

$$\begin{tikzcd}
X \arrow[d, "\alpha"'] & R \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "\sigma"] & Y \arrow[d, "\beta"] \\
FX & \sqsubseteq; (FR)^\dagger; \sqsubseteq \arrow[l, "Fp_{1\sqsubseteq}"] \arrow[r, "Fp_{2\sqsubseteq}"'] & FY
\end{tikzcd}$$

At the moment we have limited the discussion to the category of sets and we are talking about the powerset functor. We know that σ is unique in the following diagram:

$$\begin{tikzcd}
X \arrow[d, "\alpha"'] & R \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, dashed, "\sigma"] & Y \arrow[d, "\beta"] \\
\mathcal{P}X & \subseteq; (\mathcal{P}R)^\dagger; \subseteq \arrow[l, "\mathcal{P}p_{1\subseteq}"] \arrow[r, "\mathcal{P}p_{2\subseteq}"'] & \mathcal{P}Y
\end{tikzcd}$$

It is defined as $\sigma(x_1, x_2) = (\alpha(x_1), \beta(x_2))$. But σ' in the following diagram is not unique:

$$\begin{tikzcd}
X \arrow[d, "\alpha"'] \arrow[dr, phantom, "\subseteq"] & R \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "\sigma'"] \arrow[dr, phantom, "\subseteq"] & Y \arrow[d, "\beta"] \\
\mathcal{P}X & (\mathcal{P}R)^\dagger \arrow[l, "\mathcal{P}p_1^\dagger"] \arrow[r, "\mathcal{P}p_2^\dagger"'] & \mathcal{P}Y
\end{tikzcd}$$

Because assuming we have σ, for every given σ' we can define a $\delta\colon (\mathcal{P}R)^\dagger \to \subseteq; (\mathcal{P}R)^\dagger; \subseteq$ that $\sigma = \delta \cdot \sigma'$, i.e., the following diagram commutes:

$$\begin{tikzcd}
X \arrow[d, "\alpha"'] & R \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "\sigma'"] & Y \arrow[d, "\beta"] \\
\mathcal{P}X \arrow[d, "id"'] & (\mathcal{P}R)^\dagger \arrow[d, "\delta"] & \mathcal{P}Y \arrow[d, "id"] \\
\mathcal{P}X & \subseteq; (\mathcal{P}R)^\dagger; \subseteq \arrow[l, "\mathcal{P}p_{1\subseteq}"] \arrow[r, "\mathcal{P}p_{2\subseteq}"'] & \mathcal{P}Y
\end{tikzcd}$$

To define δ, we define $c\colon (\mathcal{P}R^\dagger) \to ((\mathcal{P}R^\dagger) \times R) + (\mathcal{P}R)^\dagger$ and $u\colon ((\mathcal{P}R^\dagger) \times R) + (\mathcal{P}R)^\dagger \to \subseteq; (\mathcal{P}R)^\dagger; \subseteq$ and then we define $\delta = u \cdot c$. Here are the definitions for c

and u:

$$c(w) = \begin{cases} \mathsf{inl}(w,(x_1,x_2)) & \exists x_1, x_2, \sigma'(x_1,x_2) = w \\ \mathsf{inr}\, w & \mathsf{o.w} \end{cases}$$

$$u(\mathsf{inl}\, w,(x_1,x_2)) = (\alpha(x_1), \alpha(x_2))$$

$$u(\mathsf{inr}\, w) = w$$

4.3 Symmetric simulation

Notation 4.2. From now on, we show relations with small letters, and for two relations r_1 and r_2 by $r_1 \leqslant r_2$ we mean $r_1 \subseteq r_2$. Also, we show the category of relations over set that we represent by spans with **Span**, and **Rel** is the category of sets and binary relations between them.

{lem:rel-span-equiv}

Lemma 4.3. *$r\colon X \twoheadrightarrow Y$ is a morphism in* **Rel** *iff there is an object (r, p_1, p_2) in* **Span**.

Proof. ($\Rightarrow$): $r\colon X \twoheadrightarrow Y$ being a morphism in **Rel** means that in **Set** there exist an object r with a unique mono of type $r \to X \times Y$ that is a pairing that we show with $\langle p_1, p_2\rangle$. So, (r, p_1, p_2) form an object in **Span**.

($\Leftarrow$): If (r, p_1, p_2) is an object in **Span**, then r is a binary relation from X to Y so, it is a morphism of type $X \twoheadrightarrow Y$ in **Rel**. ◻

The above translation seems to be true in a more general case, where **Span** and **Rel** are defined on an arbitrary category (the latter is called an allegory then).

Definition 4.4 (Relator). Assuming F is a functor on **Set**, a F-relator or simply a relator $\mathbf{R}$ is a monotone map that sends a morphism of **Rel** that is a relation $X \twoheadrightarrow Y$ to $FX \twoheadrightarrow FY$.

Definition 4.5 (Hermida-Jacobs Simulation). For a relator $\mathbf{R}$ on a functor F a HJ-simulation from a coalgebra $\alpha\colon X \to FX$ to a coalgebra $\beta\colon Y \to FY$ is a relation r for which there exists a morphism $\sigma\colon r \to \mathbf{R}r$ called *witness* such that the following diagram commutes (; is the relation composition): {def:hej-sim}

$$\begin{tikzcd} X \arrow[d, "\alpha"'] & r \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "\sigma"] & Y \arrow[d, "\beta"] \\ FX & \mathbf{R}r \arrow[l, "(Fp_1)^{\mathbf{R}}"] \arrow[r, "(Fp_2)^{\mathbf{R}}"'] & FY \end{tikzcd} \tag{9}$$

{eq:hej-sim}

Definition 4.6 (Relator-based Simulation). Given a relator $\mathbf{R}$, a relation $r\colon X \twoheadrightarrow Y$ is a $\mathbf{R}$-simulation from a coalgebra $\alpha\colon X \to FX$ to a coalgebra $\beta\colon Y \to FY$ if $r \leqslant \alpha; \mathbf{R}r; \beta^{\mathsf{op}}$, i.e, if $(x, y) \in r$ entails $(\alpha(x), \beta(y)) \in \mathbf{R}r$, for all $x \in X$ and $y \in Y$.

Definition 4.7 (Symmetric Relator). A relator $\mathbf{R}$ is symmetric if and only if for every relation r we have $\mathbf{R}(r^{\mathsf{op}}) = (\mathbf{R}r)^{\mathsf{op}}$.

Definition 4.8 (Relator-based Bisimulation). Given a relator $\mathbf{R}$, a relation $r\colon X \twoheadrightarrow Y$ is a $\mathbf{R}$-bisimulation from a coalgebra $\alpha\colon X \to FX$ to a coalgebra $\beta\colon Y \to FY$ if r is a $\mathbf{R}$-simulation, and $\mathbf{R}$ is a symmetric relator.

Notation 4.9. From now on we refer to relator-based simulations of a relator $\mathbf{R}$ with $\mathbf{R}$-simulation. If we talk about a HJ-simulation we specify the witness.

{lem:sim-simp}

Lemma 4.10. *r being a $\mathbf{R}$-simulation means that assuming $x\ r\ y$ we have $\alpha(x)\ \mathbf{R}r\ \beta(y)$.*

Proof. r being a $\mathbf{R}$-simulation means $r \leqslant \beta^{\mathsf{op}} \cdot \mathbf{R}r \cdot \alpha$, meaning that if $x\ r\ y$ then $x\ \alpha; \mathbf{R}r; \beta^{\mathsf{op}}\ y$, and it means that there exist $w \in \mathbf{R}r$ that its first element is equal with $\alpha(x)$ and its second element is equal with $\beta(y)$, enabling us to say $\alpha(x)\ \mathbf{R}r\ \beta(y)$. ◻

Proposition 4.11. *For a relator $\mathbf{R}$ of a functor F, a relation r is a $\mathbf{R}$-simulation from $\alpha\colon X \to FX$ to $\beta\colon Y \to FY$ iff it is a HJ-simulation with the witness $\sigma\colon r \to \mathbf{R}r$.*

Proof. ($\Rightarrow$): r being a $\mathbf{R}$-simulation means that $x\ r\ y$ gives $\alpha(x)\ \mathbf{R}r\ \beta(y)$. Since r and $\mathbf{R}r$ are both relations, by Lemma 4.3 there exist objects (r, p_1, p_2) and $(\mathbf{R}r, (Fp_1)^{\mathbf{R}}, (Fp_2)^{\mathbf{R}})$ in **Span**. We define $\sigma\colon r \to \mathbf{R}r$ to be $\sigma(x, y) = (\alpha(x), \beta(y))$. σ commutes in (9), so we have a HJ-simulation.

($\Leftarrow$): Assuming we have a σ that commutes in (9), we want to prove that if $x\ r\ y$ we have $\alpha(x)\ \mathbf{R}r\ \beta(y)$. By (9) we have $\alpha \cdot p_1 = (Fp_1)^{\mathbf{R}} \cdot \sigma$ and $\alpha \cdot p_2 = (Fp_2)^{\mathbf{R}} \cdot \sigma$. It means that since $x\ r\ y$ then exists $w \in \mathbf{R}r$, such that $\sigma(x, y) = w$, where $(Fp_1)^{\mathbf{R}}(w) = \alpha(x)$ and $(Fp_2)^{\mathbf{R}}(w) = \beta(y)$. So, we have $\alpha(x)\ \mathbf{R}r\ \beta(y)$. ◻

Proposition 4.12. *For an arbitrary relator $\mathbf{R}$ on a functor F, if a relation r is a HJ-simulation, the witness is unique.*

Proof. It only relies on the fact that $(Fp_1)^{\mathbf{R}}$ and $(Fp_2)^{\mathbf{R}}$ in (4.5) are jointly monic. ◻

Definition 4.13. We call $\hat{\mathbf{R}}$ a symmetrization of a relator $\mathbf{R}$ iff for a relation r it is defined as follows:

$$\hat{\mathbf{R}}r = \mathbf{R}r \cap (\mathbf{R}(r^{\mathsf{op}}))^{\mathsf{op}}$$

Proposition 4.14. *For every relator $\mathbf{R}$, $\hat{\mathbf{R}}$ is a relator.*

Proof. Almost obvious! ◻

Proposition 4.15. *Assuming that $\mathbf{R}$ is a relator, and r and r^{op} are both $\mathbf{R}$-simulations from a coalgebra $\alpha\colon X \to FX$ to a coalgebra $\beta\colon Y \to FY$ and vice-versa respectively, then r is also a $\hat{\mathbf{R}}$-simulation.*

Proof. We need to prove that $x\ r\ y$ gives $\alpha(x)\ \hat{\mathbf{R}}r\ \beta(y)$. r being a $\mathbf{R}$-simulation means that assuming $x\ r\ y$ we have $\alpha(x)\ \mathbf{R}r\ \beta(y)$. So, we are left to prove $\alpha(x)\ (\mathbf{R}(r^{\mathsf{op}}))^{\mathsf{op}}\ \beta(y)$. $x\ r\ y$ gives $y\ r^{\mathsf{op}}\ x$, and r^{op} being a $\mathbf{R}$-simulation from β to α gives $\beta(y)\ \mathbf{R}r^{\mathsf{op}}\ \alpha(x)$. So, we have $\alpha(x)\ (\mathbf{R}(r^{\mathsf{op}}))^{\mathsf{op}}\ \beta(y)$. ◻

Corollary 4.16. *Assuming that* $\mathbf{R}$ *is a relator, and* r *is a symmetric* $\mathbf{R}$*-simulation from a coalgebra* $\alpha\colon X \to FX$ *to itself, then* r *is also a* $\hat{\mathbf{R}}$*-simulation.*

Remark 4.17. If we want to have the previous corollary for two different coalgebras $\alpha\colon X \to FX$ and $\beta\colon X \to FX$, we need to assume that r^{op} is also a $\mathbf{R}$-simulation from β to α.

Proposition 4.18. $\hat{\mathbf{R}}$ *is a symmetric relator, i.e., every* $\hat{\mathbf{R}}$*-simulation is actually a* $\hat{\mathbf{R}}$*-bisimulation.*

Proof.

$$\begin{aligned}
&\hat{\mathbf{R}}(r^{\mathsf{op}})\\
&\quad = \mathbf{R}(r^{\mathsf{op}}) \cap (\mathbf{R}(r^{\mathsf{op}})^{\mathsf{op}})^{\mathsf{op}}\\
&\quad = \mathbf{R}(r^{\mathsf{op}}) \cap (\mathbf{R}r)^{\mathsf{op}}\\
&\quad = (\mathbf{R}r)^{\mathsf{op}} \cap \mathbf{R}(r^{\mathsf{op}})\\
&\quad = (((\mathbf{R}r)^{\mathsf{op}} \cap \mathbf{R}(r^{\mathsf{op}}))^{\mathsf{op}})^{\mathsf{op}}\\
&\quad = (\mathbf{R}r \cap (\mathbf{R}(r^{\mathsf{op}}))^{\mathsf{op}})^{\mathsf{op}}\\
&\quad = (\hat{\mathbf{R}}r)^{\mathsf{op}}
\end{aligned}$$

□

Proposition 4.19. *Assuming that* $\mathbf{R}$ *is a symmetric relator, and* r *is a* $\mathbf{R}$*-simulation from a coalgebra* (X, α) *to itself, then* r_s^{op} *is a* $\mathbf{R}$*-simulation as well.*

Proof. It is easy to directly show that r^{op} is a $\mathbf{R}$-simulation. □

Corollary 4.20. *Assuming that* $\mathbf{R}$ *is a symmetric relator, then the* $\mathbf{R}$*-simularity from a coalgebra* (X, α) *to itself is a symmetric relation.*

Proposition 4.21. *Assuming that* $\mathbf{R}$ *is a symmetric relator, then for every* r *that is a* $\mathbf{R}$*-simulation,* r^{op} *is also a* $\mathbf{R}$*-simulation.*

Proof. Assuming $y \ r^{\mathsf{op}} x$ we have $x \ r \ y$. Since r is $\mathbf{R}$-simulation we have $\alpha(x) \ \mathbf{R}r \ \beta(y)$. So, we have $\beta(y) \ (\mathbf{R}r)^{\mathsf{op}} \ \alpha(x)$, and since $\mathbf{R}$ is symmetric we have $\beta(y) \ \mathbf{R}r^{\mathsf{op}} \ \alpha(x)$. □

Definition 4.22 (Behavioural Equivalence). Two states x and y of two coalgebras (X, α) and (Y, β) are behaviourally equivalent iff there exist a coalgebra (Z, γ) and coalgebra morphisms $f\colon (X, \alpha) \to (Z, \gamma)$ and $g\colon (Y, \beta) \to (Z, \gamma)$ such that $f(x) = g(y)$. The relation r consisting of all behaviourally equivalent states of these two coalgebras is called behavioural equivalence.

Definition 4.23 (Difunctional Relation). A relation $r\colon X \twoheadrightarrow Y$ is difunctional iff there are functions $f\colon X \to Z$ and $g\colon Y \to Z$ such that for every $(x, y) \in R$ we have $f(x) = g(y)$.

Definition 4.24 (Soundness and Completeness of R-similarity). For a relator $\mathbf{R}$ the $\mathbf{R}$-similarity from a coalgebra (X, α) to a coalgebra (Y, β) is sound iff it is less than or equal to their behavioural equivalence, and it is complete iff it is greater than or equal to their behavioural equivalence.

Theorem 4.25. *Let $\mathbf{R}$ be a relator for a functor F:*

1. *If for all functions $f\colon X \to A$ and $g\colon Y \to A$, $\mathbf{R}(g^{\mathsf{op}} \cdot f) \geqslant (Fg)^{\mathsf{op}} \cdot Ff$, then $\mathbf{R}$-similarity is complete.*
2. *If $\mathbf{R}$ preserves difunctional relations and for every epi-cospan $(f\colon X \to A, g\colon Y \to A) \in \mathbf{Set}$, $\mathbf{R}(g^{\mathsf{op}} \cdot f) \leqslant (Fg)^{\mathsf{op}} \cdot Ff$, then $\mathbf{R}$-similarity is sound.*

□

{prop:difunc-preser}

Proposition 4.26. *Assuming that $\mathbf{R}$ is a F-relator that preserves difunctional relations, then $\hat{\mathbf{R}}$ does the same.*

Proof. Assuming r is difunctional, then there exist $f_1, f_2\colon FX \to FZ$ and $g_1, g_2\colon FY \to FZ$ such that $p\ \mathbf{R}r\ q$ iff $f_1(p) = g_1(q)$ and $q\ \mathbf{R}r^{\mathsf{op}}\ p$ iff $f_2(p) = g_2(q)$. By the definition of $\hat{\mathbf{R}}$ we have $p\ \hat{\mathbf{R}}r\ q$ iff $p\ \mathbf{R}r\ q$ and $p\ (\mathbf{R}r^{\mathsf{op}})^{\mathsf{op}}\ q$ that is equivalent to say that $\langle f_1, f_2\rangle(p) = \langle g_1, g_2\rangle(q)$. □

Corollary 4.27. *Assuming that $\mathbf{R}$ is a F-relator that preserves difunctional relations, for every symmetric relation r we have $\hat{\mathbf{R}}r = \mathbf{R}r$.*

Proof. $\mathbf{R}$-similarity being symmetric means that for the f_1, f_2, g_1 and g_2 in the proof of Proposition 4.26, we have $f_1 = f_2$ and $g_1 = g_2$. □

Assuming that $\mathbf{R}$-similarity is complete, does not guarantee that $\hat{\mathbf{R}}$-similarity is sound and complete. We give a counter-example. Assuming that $\mathbf{R}$ is a $\mathcal{P}$-relator that takes $r\colon X \twoheadrightarrow Y$ to $\mathcal{P}X \times \mathcal{P}Y$, then $\hat{\mathbf{R}}r = \mathcal{P}X \times \mathcal{P}Y$ as well. It means that for every coalgebras (X, α), (Y, β), $\hat{\mathbf{R}}$-similarity is equal to $X \times Y$, which is rare to be equal to behavioural equivalence. For example, if we take $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, y_3\}$, and we define α and β as

$$\alpha(x) = \begin{cases} \{x_1, x_2\} & x = x_1 \\ \{x_2\} & x = x_2 \end{cases}$$

and

$$\beta(y) = \begin{cases} \{y_3\} & y = y_1 \\ Y & y = y_2 \\ \emptyset & y = y_3 \end{cases}$$

then at least (x_1, y_3) is not in the behavioural equivalence, while it is in $\hat{\mathbf{R}}$-similarity.

Proposition 4.28. *Assuming that $\mathbf{R}$-similarity is symmetric and complete, then $\hat{\mathbf{R}}$-similarity from a coalgebra $\alpha\colon X \to FX$ to itself is sound and complete.*

Proof. (Completeness): We show $\mathbf{R}$-similarity with r_s and $\hat{\mathbf{R}}$-similarity with $r_{\hat{s}}$. Also, we show the behabioural equivalence with r_b. Since

Proposition 4.29. *Assuming that* $\mathbf{R}$ *is a* F*-relator (*F *is a set functor), that for every functions* $f\colon X \to Z$ *and* $g\colon Y \to Z$*, we have* $\mathbf{R}(g^{\mathsf{op}} \cdot f) \geqslant (Fg)^{\mathsf{op}} \cdot Ff$*, then* $\hat{\mathbf{R}}$*-similarity is complete.*

Proof. We need to prove that for every functions $f\colon X \to Z$ and $g\colon Y \to Z$ we have $\hat{\mathbf{R}}(g^{\mathsf{op}} \cdot f) \geqslant (Fg)^{\mathsf{op}} \cdot Ff$. By the assumption we have $\mathbf{R}(g^{\mathsf{op}} \cdot f) \geqslant (Fg)^{\mathsf{op}} \cdot Ff$. Also, again from the assumption we have $\mathbf{R}(f^{\mathsf{op}} \cdot g) \geqslant (Ff)^{\mathsf{op}} \cdot Fg$ that gives $(\mathbf{R}(f^{\mathsf{op}} \cdot g))^{\mathsf{op}} \geqslant (Fg)^{\mathsf{op}} \cdot Ff$. So, we have $\hat{\mathbf{R}}(g^{\mathsf{op}} \cdot f) \geqslant F(g)^{\mathsf{op}} \cdot Ff$. ◻

Proposition 4.30. *Assuming that* $\mathbf{R}$ *is a symmetric relator for a functor* $F\colon \mathbf{Set} \to \mathbf{Set}$*, then the* $\mathbf{R}$*-bisimilarity from a coalgebra* $\alpha\colon X \to FX$ *to itself is sound, using the axiom of choice.*

Proof. We call the bisimilarity relation r, and we assume $x_1 \; r \; x_2$, now we need to prove that x_1 and x_2 are behaviourally equivalent. We take $Z = X/r$, where $X/r = \{[x] \mid [x] = \{y \mid x \; r \; y\}\}$. Now, we define the coalgebra homomorphism $f\colon X \to X/r$ as $f(x) = [x]$. So, assuming $x_1 \; r \; x_2$ gives $f(x_1) = f(x_2)$. Now, assuming that exists a choice function $c\colon X/r \to X$ that $c \cdot f = \mathsf{id}_X$, we define $\gamma\colon X/r \to F(X/r)$, as $\gamma([x]) = Ff \cdot \alpha \cdot c([x])$. Now, we have

$$
\begin{aligned}
&\gamma \cdot f \\
&\qquad = Ff \cdot \alpha \cdot c \cdot f \\
&\qquad = Ff \cdot \alpha.
\end{aligned}
$$

So, x_1 and x_2 are behaviourally equivalent. So, the $\mathbf{R}$-bisimilarity is sound. ◻

Corollary 4.31. *Assuming that a relator* $\mathbf{R}$ *over a functor* $F\colon \mathbf{Set} \to \mathbf{Set}$ *satisfies* $\mathbf{R}(g^{\mathsf{op}} \cdot f) \geqslant (Fg)^{\mathsf{op}} \cdot Ff$ *for every functions* $f\colon X \to Z$ *and* $g\colon Y \to Z$*, then* $\hat{\mathbf{R}}$*-bisimilarity from a coalgebra* $\alpha\colon X \to FX$ *to itself is sound and complete, using the axiom of choice.*

4.4 Egli-Milner relator and Barr relators

Definition 4.32. We call the map $\mathbf{L}\colon \mathbf{Rel} \to \mathbf{Rel}$ the Egli-Milner $\mathcal{P}$-relator, whenever for every relation $r\colon X \twoheadrightarrow Y$ it is defined as follows:

$$\mathbf{L}r = \{(S,T) \mid x \in S \Rightarrow \exists y \in T, x \; r \; y\}$$

Egli-Milner relator is not sound or complete, although its symmetrization is sound and complete.

Proposition 4.33. $\hat{\mathbf{L}}$*-similarity from a coalgebra* (α, X) *to* (β, Y) *is sound and complete.*

Proof. We need to prove that for every functions $f\colon X \to Z$ and $g\colon Y \to Z$, $\hat{\mathbf{L}}(g^{\mathsf{op}}\cdot f) = (\mathcal{P}g)^{\mathsf{op}}\cdot\mathcal{P}f$. We have $S\ \hat{\mathbf{L}}(g^{\mathsf{op}}\cdot f)\ T$ iff $S\ \mathbf{L}(g^{\mathsf{op}}\cdot f)\ T$ and $T\ \mathbf{L}(f^{\mathsf{op}}\cdot g)\ S$. Then we have

$$\begin{aligned}
&S\ \mathbf{L}(g^{\mathsf{op}}\cdot f)\ T\\
&\qquad\iff \forall x \in S, \exists y \in T, x\ g^{\mathsf{op}}\cdot f\ y\\
&\qquad\iff \forall x \in S, \exists y \in T, z \in Z, x\ f\ z\ ,\ y\ g\ z,
\end{aligned}$$

and

$$\begin{aligned}
&T\ \mathbf{L}(f^{\mathsf{op}}\cdot g)\ S\\
&\qquad\iff \forall y \in T, \exists x \in S, y\ f^{\mathsf{op}}\cdot g\ x\\
&\qquad\iff \forall y \in T, \exists x \in S, z \in Z, x\ f\ z\ ,\ y\ g\ z.
\end{aligned}$$

It is equivalent with the following:

$$\begin{aligned}
&\forall x \in S, \exists y \in T, f(x) = g(y),\\
&\forall y \in T, \exists x \in S, f(x) = g(y).
\end{aligned}$$

Equivalently, $Im(f\mid_S) = Im(g\mid_T)$, and we call images U that is in $\mathcal{P}Z$. So, we equivalently have

$$\begin{aligned}
&S\ \mathcal{P}f\ U,\ T\ \mathcal{P}g\ U\\
&\qquad\iff S\ \mathcal{P}f\ U,\ U\ (\mathcal{P}g)^{\mathsf{op}}\ T\\
&\qquad\iff S\ (\mathcal{P}g)^{\mathsf{op}}\cdot\mathcal{P}f\ T
\end{aligned}$$

□

For every relation $r \twoheadrightarrow X \to Y$ $\mathbf{L}r =\subseteq\ \mathbf{L}r = \mathbf{L}r\ \subseteq=\subseteq; \mathbf{L}r; \subseteq$.

Barr relator is a generalization of the Egli-Milner relator, where the functor is generalized.

Definition 4.34 (Barr relator). A relator over a functor F is a Barr relator, shown by $\bar{F}$, iff for a relation $r\colon X \twoheadrightarrow Y$, and a span $(\pi_1\colon A \to X, \pi_2\colon A \to Y)$ that $r = \pi_2\cdot\pi_1^{\mathsf{op}}$ we have:

$$\bar{F}r = F\pi_2\cdot(F\pi_1)^{\mathsf{op}}$$

Proposition 4.35. *For every set-functor F, the barr relator $\bar{F}$ is symmetric.*

Proof. Assuming that for a span $(\pi_1\colon A \to X, \pi_2\colon A \to Y)$ we have $r = \pi_2\cdot\pi_1^{\mathsf{op}}$, then we have:

$$\begin{aligned}
&\hat{\bar{F}}r\\
&=\bar{F}r \cap (\bar{F}r^{\mathsf{op}})^{\mathsf{op}}\\
&=\bar{F}(\pi_2\cdot\pi_1^{\mathsf{op}}) \cap (\bar{F}(\pi_2\cdot\pi_1^{\mathsf{op}})^{\mathsf{op}})^{\mathsf{op}}\\
&=\bar{F}(\pi_2\cdot\pi_1^{\mathsf{op}}) \cap (\bar{F}(\pi_1\cdot\pi_2^{\mathsf{op}}))^{\mathsf{op}}
\end{aligned}$$

$$
\begin{aligned}
&= F\pi_2 \cdot (F\pi_1)^{\mathsf{op}} \cap (F\pi_1 \cdot (F\pi_2)^{\mathsf{op}})^{\mathsf{op}} \\
&= F\pi_2 \cdot (F\pi_1)^{\mathsf{op}} \cap F\pi_2 \cdot (F\pi_1)^{\mathsf{op}} \\
&= F\pi_2 \cdot (F\pi_1)^{\mathsf{op}} \\
&= \bar{F} r
\end{aligned}
$$

□

Proposition 4.36. *$\hat{L}$ is a Barr relator.*

Proof. TODO: Finish.

Definition 4.37 (Mid-lax Barr relator). Given a relation r, and take a span $(\pi_1 \colon A \to X, \pi_2 \colon A \to Y)$ that $r = \pi_2 \cdot \pi_1^{\mathsf{op}}$. Assuming that $\sqsubseteq$ is a partial order over a functor F, then the relator over F and shown with $\overrightarrow{F}$ is a *mid-lax Barr relator* if we have:

$$\overrightarrow{F} r = F\pi_2 \cdot \sqsubseteq \cdot (F\pi_1)^{\mathsf{op}}$$

Proposition 4.38. *For every functor $F \colon \mathbf{Set} \to \mathbf{Set}$, the symmetrization of the mid-lax Bar relator is equal with the Barr relator.*

Proof. TODO: Finish.

Proposition 4.39. *For a span $(\pi_1 \colon A \to X, \pi_2 \colon A \to Y)$ the following propositions hold:*

1. $\mathcal{P}\pi_2 \cdot \subseteq \cdot (\mathcal{P}\pi_1)^{\mathsf{op}} \quad = \quad \subseteq \cdot \mathcal{P}\pi_2 \cdot (\mathcal{P}\pi_1)^{\mathsf{op}}$
2. $\mathcal{P}\pi_1 \cdot \subseteq \cdot (\mathcal{P}\pi_2)^{\mathsf{op}} \quad = \quad \subseteq \cdot \mathcal{P}\pi_1 \cdot (\mathcal{P}\pi_2)^{\mathsf{op}}$

Proof. Without loss of generality, we assume $i, j \in \{1, 2\}$, and $i \neq j$, and prove $\mathcal{P}\pi_j \cdot \subseteq \cdot (\mathcal{P}\pi_i)^{\mathsf{op}} \quad = \quad \subseteq \cdot \mathcal{P}\pi_j \cdot (\mathcal{P}\pi_i)^{\mathsf{op}}$.

Assuming $x \; \mathcal{P}\pi_j \cdot \subseteq \cdot (\mathcal{P}\pi_i)^{\mathsf{op}} \; y$, then exist z and z' such that

$$
\begin{gathered}
z \; (\mathcal{P}\pi_i) \; x, \\
z \subseteq z', \\
z' \; (\mathcal{P}\pi_j) \; y.
\end{gathered}
$$

Then from $z \subseteq z'$ we get $\mathcal{P}\pi_j(z) \subseteq y$. So, we have $z \subseteq \cdot \mathcal{P}\pi_j \; y$, thus $x \subseteq \cdot \mathcal{P}\pi_j \cdot (\mathcal{P}\pi_i)^{\mathsf{op}} \; y$.

Now, assuming $x \subseteq \cdot \mathcal{P}\pi_j \cdot (\mathcal{P}\pi_i)^{\mathsf{op}} \; y$, then there exist z and y' such that

$$
\begin{gathered}
z \; (\mathcal{P}\pi_i) \; x, \\
z \; (\mathcal{P}\pi_j) \; y', \\
y' \subseteq y.
\end{gathered}
$$

We take the set $w = z \cup (\mathcal{P}\pi_i(z) \times y)$ for which we have $z \subseteq w$ and $\mathcal{P}\pi_j(w) = y$. So, we have $w \; (\mathcal{P}\pi_j) \; y$, $z \subseteq w$, and $z \; (\mathcal{P}\pi_i) \; x$ that gives $x \; \mathcal{P}\pi_j \cdot \subseteq \cdot (\mathcal{P}\pi_i)^{\mathsf{op}} \; y$. □

Proposition 4.40. *For a span $(\pi_1\colon A \to X, \pi_2\colon A \to Y)$, assuming that F is either the maybe monad, the subdistribution monad or $FX = \mathcal{P}(X \times A)$, where A is a set of labels, then the following propositions hold:*

1. $F\pi_2 \cdot \sqsubseteq \cdot (F\pi_1)^{\mathsf{op}} \quad = \quad \sqsubseteq \cdot F\pi_2 \cdot (F\pi_1)^{\mathsf{op}}$
2. $F\pi_1 \cdot \sqsubseteq \cdot (F\pi_2)^{\mathsf{op}} \quad = \quad \sqsubseteq \cdot F\pi_1 \cdot (F\pi_2)^{\mathsf{op}}$

Proof. TODO: Finish.

{def:nat-ord}

Definition 4.41 (Natural Order Structure). A *natural order structure* on a functor F is a preorder $\sqsubseteq$ on each Hom-set of the form $\mathsf{Hom}(X, FY)$ such that if $\alpha \sqsubseteq \beta$ in $\mathsf{Hom}(X, FY)$, $f\colon X \to X'$, $g\colon Y \to Y'$, then:

{item:nat-ord:I} (I) $\alpha \cdot f \sqsubseteq \beta \cdot f$ in $\mathsf{Hom}(X', Y)$.
{item:nat-ord:II} (II) $Fg \cdot \alpha \sqsubseteq Fg \cdot \beta$ in $\mathsf{Hom}(X, Y')$.

{def:good-ord}

Definition 4.42 (Good Order Structure). A *good order structure* on a functor F is a preorder $\sqsubseteq$ on each Hom-set of the form $\mathsf{Hom}(X, FY)$ that is a natural order structure, and if $h\colon X \to FZ$, $k\colon X \to FY$, $g\colon Y \to Z$, $h \sqsubseteq Fg \cdot k$ in $\mathsf{Hom}(X, FZ)$, then there is $k'\colon X \to FY$ such that $k' \sqsubseteq k$ in $\mathsf{Hom}(X, FY)$ and $h = Fg \cdot k'$.

{def:cogood-ord}

Definition 4.43 (Cogood Order Structure). A *cogood order structure* on a functor F is a preorder $\sqsubseteq$ on each Hom-set of the form $\mathsf{Hom}(X, FY)$ that is a natural order structure, and if $h\colon X \to FZ$, $k\colon X \to FY$, $g\colon Y \to Z$, $Fg \cdot k \sqsubseteq h$ in $\mathsf{Hom}(X, FZ)$, then there is $k'\colon X \to FY$ such that $k \sqsubseteq k'$ in $\mathsf{Hom}(X, FY)$ and $h = Fg \cdot k'$.

{lem:good}

Lemma 4.44. *Assuming that a a functor F has an order structure $\sqsubseteq$ that is good, then for every $f \in \mathsf{Hom}(X, FY)$ we have:*

(I) $Ff \cdot \sqsupseteq \quad = \quad \sqsupseteq \cdot Ff$
(II) $(Ff)^{\mathsf{op}} \cdot \sqsubseteq \quad = \quad \sqsubseteq \cdot (Ff)^{\mathsf{op}}$

Proof. (I) Assuming $t\ Ff \cdot \sqsupseteq x$, there exists s such that $t \sqsupseteq s$ and $Ff(s) = x$. Since $\sqsubseteq$ is good, and thus natural, by Definition 4.41, from $s \sqsubseteq t$ we get $x \sqsubseteq Ff(t)$ that is $t \sqsupseteq \cdot Ff\ x$.

Assuming $t \sqsupseteq \cdot Ff\ x$, there exists y such that $Ff(t) = y$ and $y \sqsupseteq x$. By Definition 4.42 since $Ff(t) \sqsupseteq x$ there exists s that $t \sqsupseteq s$ and $Ff(s) = x$ that is $t\ Ff \cdot \sqsubseteq s$.

(II) Basically, by definition of op and relation composition we have

$$(Ff \cdot \sqsubseteq)^{\mathsf{op}} = \sqsupseteq \cdot (Ff)^{\mathsf{op}},$$
$$(\sqsubseteq \cdot Ff)^{\mathsf{op}} = (Ff)^{\mathsf{op}} \cdot \sqsupseteq.$$

So it follows directly from applying op on both sides of (I). □

{lem:cogood}

Lemma 4.45. *Assuming that a a functor F has an order structure $\sqsubseteq$ that is cogood, then for every $f \in \mathsf{Hom}(X, FY)$ we have:*

(I) $Ff \cdot \sqsubseteq \quad = \quad \sqsubseteq \cdot Ff$

(II) $(Ff)^{\mathsf{op}} \cdot \sqsupseteq \quad = \quad \sqsupseteq \cdot (Ff)^{\mathsf{op}}$

Proof. (I) Assuming $t\ Ff \cdot \sqsubseteq x$, there exists s such that $t \sqsubseteq s$ and $Ff(s) = x$. Since $\sqsubseteq$ is cogood, and thus natural, by Definition 4.41, from $t \sqsubseteq s$ we get $Ff(t) \sqsubseteq x$ that is $t \sqsubseteq \cdot Ff\ x$.

Assuming $t \sqsubseteq \cdot Ff\ x$, there exists y such that $Ff(t) = y$ and $y \sqsubseteq x$. By Definition 4.43 since $Ff(t) \sqsubseteq x$ there exists s that $t \sqsubseteq s$ and $Ff(s) = x$ that is $t\ Ff \cdot \sqsubseteq s$.

(II) Basically, by definition of op and relation composition we have

$$(Ff \cdot \sqsubseteq)^{\mathsf{op}} = \sqsupseteq \cdot (Ff)^{\mathsf{op}},$$
$$(\sqsubseteq \cdot Ff)^{\mathsf{op}} = (Ff)^{\mathsf{op}} \cdot \sqsupseteq .$$

So it follows directly from applying op on both sides of (I). ◻

Proposition 4.46. *For a span $(\pi_1 \colon A \to X, \pi_2 \colon A \to Y)$, assuming that F has an order structure $\sqsubseteq$, the following propositions hold:*

1. *If $\sqsubseteq$ is good, we have* $F\pi_2 \cdot \sqsubseteq \cdot (F\pi_1)^{\mathsf{op}} \quad = \quad F\pi_2 \cdot (F\pi_1)^{\mathsf{op}} \cdot \sqsubseteq$
2. *If $\sqsubseteq$ is cogood, we have* $F\pi_2 \cdot \sqsubseteq \cdot (F\pi_1)^{\mathsf{op}} \quad = \quad \sqsubseteq \cdot F\pi_2 \cdot (F\pi_1)^{\mathsf{op}}$
3. *If $\sqsubseteq$ is both good and cogood, all the following are equal:*
   - $F\pi_2 \cdot \sqsubseteq \cdot (F\pi_1)^{\mathsf{op}}$
   - $F\pi_2 \cdot (F\pi_1)^{\mathsf{op}} \cdot \sqsubseteq$
   - $\sqsubseteq \cdot F\pi_2 \cdot (F\pi_1)^{\mathsf{op}}$
   - $\sqsubseteq \cdot F\pi_2 \cdot (F\pi_1)^{\mathsf{op}} \cdot \sqsubseteq$

Proof. They all follow in an obvious way from Lemma 4.44 and Lemma 4.45. The last one needs $\sqsubseteq \cdot \sqsubseteq = \sqsubseteq$ that comes from transitivity of $\sqsubseteq$. ◻

Example 4.47. In the category of sets, subset over the powerset functor is an example of a good order. For every $h \colon X \to \mathcal{P}Z$, $k \colon X \to \mathcal{P}Y$, and $g \colon Y \to Z$, such that $h \subseteq \mathcal{P}g \cdot k$, there exists $k' \colon X \to \mathcal{P}Y$ that for every $x \in X$, $k'(x) = k(x) \backslash \{y \mid g(y) \notin h(x)\}$. We show that for every $x \in X$, we have $\mathcal{P}g \cdot k'(x) = h(x)$.

Assuming $z \in \mathcal{P}g(k'(x))$, then there exist $y' \in k(x) \backslash \{y \mid g(y) \notin h(x)\}$ that $g(y') = z$, so $y' \in k(x)$, $y' \notin \{y \mid g(y) \notin h(x)\}$. $y' \notin \{y \mid g(y) \notin h(x)\}$ means that $g(y') \in h(x)$, thus $\mathcal{P}g \cdot k' \subseteq h$.

Assuming $z \in h(x)$, since $h \subseteq \mathcal{P}g \cdot k$, then $z \in \mathcal{P}g \cdot k(x)$. So, there exists y' such that $y' \in k(x)$ and $g(y') = z$. Since $g(y') \in h(x)$ then $y' \notin \{y \mid g(y) \notin h(x)\}$ meaning that $y' \in k'(x)$ that means $z \in \mathcal{P}g(k'(x))$, thus $h \subseteq \mathcal{P}g \cdot k'$. ◻

Example 4.48. In the category of sets, subset over the powerset functor is NOT an example of a cogood order! For some set X we take $h \colon X \to \mathcal{P}\mathbb{Z}$, for every $x \in X$, $h(x) = \mathbb{Z}$, $g \colon \mathbb{Z} \to \mathbb{Z}$, and for every $z \in \mathbb{Z}$,

$$g(z) = \begin{cases} 0 & z \in \mathbb{Z}^+ \\ 1 & \text{otherwise} \end{cases}$$

then

$$\mathcal{P}g(A) = \begin{cases} \emptyset & A = \emptyset \\ \{0\} & A \subseteq \mathbb{Z}^+, A \neq \emptyset \\ \{1\} & A \subseteq \mathbb{Z}^- \cup \{0\}, A \neq \emptyset \\ \{0,1\} & \text{otherwise} \end{cases}$$

then no matter what $k\colon X \to \mathcal{P}\mathbb{Z}$ is there will be no $k'\colon X \to \mathcal{P}\mathbb{Z}$, for every $x \in X$, $\mathcal{P}g(k'(x)) = h(x)$, as $\mathcal{P}g(k'(x)) \subseteq \{0,1\}$, while $h(x) = \mathbb{Z}$, so $\mathcal{P}g(k'(x)) \subset h(x)$.

The previous example suggests that cogoodness is a strong condition. Perhaps it can be limited to be meaningful. Maybe the morphism g can be limited to legs of a span $(\pi_1\colon A \to X, \pi_2\colon A \to Y)$, where for a relation r, $r = \pi_2 \cdot \pi_1^{\mathsf{op}}$.

Proposition 4.49. *For a natural order structure $\sqsubseteq$ on a set-functor F, if for every $f \in \mathsf{Hom}(X, FY)$ we have $Ff \cdot \sqsubseteq = \sqsubseteq \cdot Ff$, then $\sqsubseteq$ is cogood.*

Proof. `TODO: Investigate if it's true! If it is, is naturality necessary? It is really weird! If having a good order structure forces every witness for AM-simulation to have the left cell of the lax diagram as equality, and subset on powerset functor is a good order structure, then what is the counter-example number 4 that you have?!`

Proposition 4.50. *Assuming that $\mathbf{R}$ is a difunctionally functorial relator, then the symmetrization of the relator that takes $r\colon X \twoheadrightarrow Y$ to $\mathbf{R}r \cdot \sqsubseteq$ is a sound relator.*

Proof. `TODO: Finish.`

Proposition 4.51. *Assuming that $\mathbf{R}$ is a relator over $F\colon \mathbf{Set} \to \mathbf{Set}$, and $\sqsubseteq_X$ and $\sqsubseteq_Y$ are posets over FX and FY respectively, then the symmetrization of the relator that takes $r\colon X \twoheadrightarrow Y$ to $\sqsubseteq_X; \mathbf{R}r; \sqsubseteq_Y$ is a Barr relator.*

Proof. `TODO: Finish.`