Coalgebraic Notions of Simulation, Bisimulation and Relators

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Overview

The content of the talk:

- Intuitive introduction to coalgebra and (bi)simulation
- Reviewing basic definitions of coalgebra and (bi)simulation
- Relator-based notions
- Span-based notions
- Motivation! Howe's method for categorical operational semantics

Why Does Equivalence of Systems Matter?

A basic but fundamental question

When can we say that two systems *behave the same*?

This question is everywhere, often silently assumed:

- **Hardware redesign.** A chip redesigned for cheaper production must still compute exactly what the old one did.
- </> **Software upgrades.** A bank replaces its backend — from the customer's view (balances, transactions), nothing should change.

The catch

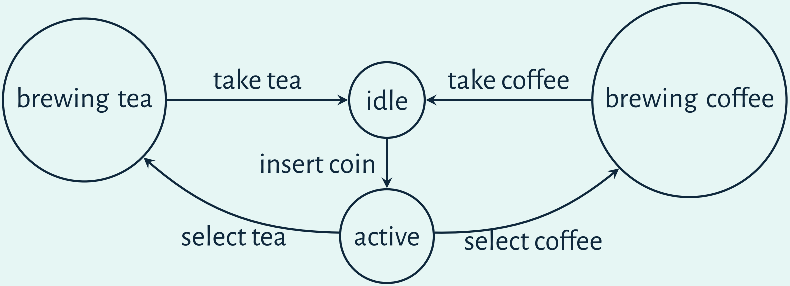
"Behaving the same" is intuitive, but making it *precise* and *checkable* is surprisingly subtle.

From "System" to Mathematics

The word "system" needs to be made precise.

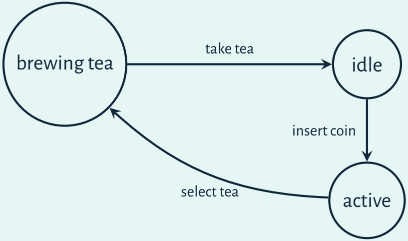
We model systems as mathematical notions like **labeled transition systems**: a set of states, together with a transition relation between them.

Example: a vending machine

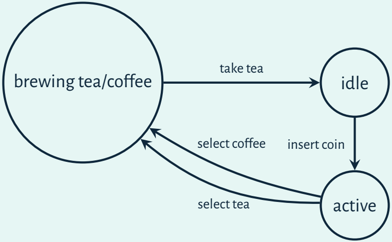


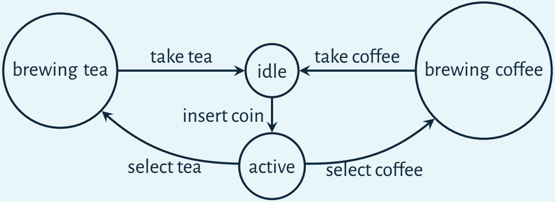
Simulation and Bisimulation

Similar to



Bisimilar with





Labelled Transition Systems

A *labelled transition system* is a triple (X, A, T), where $T \subseteq X \times A \times X$. We write $x \xrightarrow{a} x'$ for $(x, a, x') \in T$.

Simulation and Bisimulation

A relation $R \subseteq X \times Y$ is a simulation from (X, A, T) to (Y, A, S) whenever

$$x\ R\ y,\ x \xrightarrow{a} x' \Rightarrow \exists y'.\ y \xrightarrow{a} y' \ \wedge\ x' R y'.$$

A simulation relation like R is a bisimulation whenever

$$x\ R\ y,\ y \xrightarrow{a} y' \Rightarrow \exists x'.\ x \xrightarrow{a} x' \ \wedge\ x' R y'.$$

Similarity and Bisimilaritiy

The *similarity relation* $\precsim \subseteq X \times Y$ is the greatest simulation relation. Equivalently

$$x \precsim y \iff \exists R.\ xRy,$$

where R is a simulation.

Similarly, the *bisimilarity relation* $\sim \subseteq X \times Y$ is the greatest bisimulation relation. Equivalently

$$x \sim y \iff \exists R.\ xRy,$$

where R is a bisimulation.

Ultimately, people care about similarity and bisimilarity, but simulation and bisimulation should be defined first.

Coalgebras

Fix an endofunctor

$$F : \mathbf{C} \to \mathbf{C}.$$

An F-coalgebra is a pair (X, γ), consisting of an object X, and a map

$$\gamma : X \to FX.$$

Different choices of F describe different transition systems.

Our Goal

Develop **simulation, bisimulation** once, for all coalgebras.

Different kinds of systems

- Labeled Transition Systems ($FX = \mathcal{P}(A \times X)$)
- Kripke frames ($FX = \mathcal{P}X$)
- Operational Semantics of Programming Languages (*Abstract HO-GSOS, and Big-step SOS!*)
- Markov Chains ($FX = (\mathcal{D}X)^A$)
- Mealy Machines ($FX = (O \times X)^I$)

$\vdots$

Relator

Assuming $F : \textbf{Set} \to \textbf{Set}$ is a functor, an F-relator a monotone map that sends a morphism of **Rel** that is a relation $X \rightarrowtail\!\!\!\twoheadrightarrow Y$ to a relation $FX \rightarrowtail\!\!\!\twoheadrightarrow FY$.

R-Simulation

For an F-relator R, a relation $r : X \rightarrowtail\!\!\!\twoheadrightarrow Y$ is a R-simulation from a coalgebra (X, α) to a coalgebra (Y, β) if $r \subseteq \beta \cdot \mathrm{R}r \cdot \alpha^{-1}$.

Symmetrization of a Relator

Symmetrization of a relator R is defined as:

$$\hat{\mathrm{R}} = \mathrm{R}r \cap (\mathrm{R}r^{-1})^{-1}$$

Symmetric Relator

A relator R is symmetric if $\mathrm{R} = \hat{\mathrm{R}}$

R-Bisimulation

For an F-relator R, a relation $r : X \rightarrowtail\!\!\!\twoheadrightarrow Y$ is a R-bisimulation from a coalgebra (X, α) to a coalgebra (Y, β) if it is an R-simulation, and R is a symmetric relator.

R-Similarity

R-similarity is the greatest R-simulation.

R-Bisimilarity

R-bisimilarity is the greatest R-bisimulation.

Example: Trivial Relator

For an arbitrary set endofunctor F, the relator $\mathbf{Q}$ takes every relation $r : X \nrightarrow Y$ to $FX \times FY$.

It is a symmetric relator.

But it is a bad relator because every relation is a $\mathbf{Q}$-bisimulation!

Example: Egli-Milner Relator

The relator $\mathbf{L}$ is a $\mathcal{P}$-relator defined as:

$$\mathbf{L}r = \{(S, T) \mid x \in S \Rightarrow \exists y \in T, x\ r\ y\}$$

We give a family of examples:

Example: Barr Relators

Given a set endofunctor F, and r is a relation that $r = \pi_2 \cdot \pi_1^{-1}$ (π_1 and π_2 are functions) the Barr relator $\bar{F}$ is defined as $\bar{F}r = \bar{F}\pi_2 \cdot (\bar{F}\pi_1)^{-1}$.

The symmetrization of the Egli-Milner relator is a Barr relator, i.e.,

$$\hat{\mathbf{L}} = \bar{\mathcal{P}}$$

Barr relators are nicely behaved relators.

A trivial relator is a not a Barr relator.

Lax Barr relators

An ordering of a set endofunctor F is a functor G that commutes in the following diagram:

$$\begin{tikzcd} & \mathbf{PoSet} \arrow[dr, "U"] & \\ \mathbf{Set} \arrow[ur, "G"] \arrow[rr, "F"'] & & \mathbf{Set} \end{tikzcd}$$

Meaning that for every X, there is a poset $(FX, \sqsubseteq)$.

For example $(\mathcal{P}X, \subseteq)$ for every set X.

With orderings we define *lax Barr relators*. Lax Barr relators are different abstractions of the Egli-Milner relator.

There are multiple of them:

- Left-lax: $\sqsubseteq \cdot \bar{F}r$
- Right-lax: $\bar{F}\cdot \sqsubseteq$
- Bi-lax: $\sqsubseteq \cdot \bar{F}r \cdot \sqsubseteq$
- Mid-lax: $F\pi_1 \cdot \sqsubseteq \cdot (F\pi_2)^{-1}$

Liftability

An ordering on a set endofunctor F is called *liftable*, if

$$\alpha \sqsubseteq \beta \Rightarrow \alpha \cdot f \sqsubseteq \beta \cdot f,$$

$$\alpha \sqsubseteq \beta \Rightarrow Ff' \cdot \alpha \sqsubseteq Ff' \cdot \beta,$$

$$h \sqsubseteq Fg \cdot k \Rightarrow \exists k', h = Fg \cdot k \;\&\; k' \sqsubseteq k.$$

Proposition

Given a set endofunctor F, with a liftable ordering

- mid-lax and left-lax relators of F are equal,
- Assuming the axiom of choice, righ-lax and left-lax relators of F are equal,
- the symmetrizaion of the left-lax relator is *sound* and *complete* with respect to *behavioural equivalence*.

Aczel-Mendler Bisimulation

For coalgebras (X, α), and (Y, β) a span (R, X, Y) is an *AM-bisimulation*, if there is a *witness* γ that commutes in the following:

$$\begin{tikzcd} X \arrow[d, "\alpha"'] & R \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "\gamma"] & Y \arrow[d, "\beta"] \\ FX & FR \arrow[l, "Fp_1"] \arrow[r, "Fp_2"'] & FY \end{tikzcd}$$

In a *regular category*, if we change FR with $\bar{F}R$ that shows the image of $< Fp_1, Fp_2 >$, and the legs of $\bar{F}$ (Barr relator in **Set**), we have an *Hermida-Jacobs* bisimulation.

Every AM-bisimulation is an HJ-bisimulation.

Assuming the axiom of choice, every HJ-bisimulation is an AM-bisimulation.

We have *AM-simulation*, if we lax the diagram:

$$\begin{tikzcd} X \arrow[d, "\alpha"'] \arrow[dr, phantom, "\sqsubseteq"] & R \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "\sigma"] \arrow[dr, phantom, "\sqsupseteq"] & Y \arrow[d, "\beta"] \\ FX & FR \arrow[l, "Fp_1"] \arrow[r, "Fp_2"'] & FY \end{tikzcd}$$

We have found a symmetric AM-simulation that is not a bisimulation!

Proposition

Given a symmetric span (R, X), with the symmetry evidence $s : R \to R$, assuming R is an AM-simulation on a coalgebra (X, α) with evidence σ, then R is an

Motivation!

We have a coalgbera $(\mu\Sigma, \gamma)$ in an arbitrary category $\mathbb{C}$.

-$\mu\Sigma$ is representing terms of a programming language,

-and γ is giving their big-step reduction.

Main Goal

We want to prove that for terms t and s, and a context C:

$$t \sim s \Rightarrow C[t] \sim C[s]$$

-In the literature, this is often proved using *Howe's method*.

-The method, applies a closure on the bisimilarity relation on $\mu\Sigma$.

-Then one should prove that the result is a simulation.

-The closure preserves symmetry, and symmetric simulation is a bisimulation in traditional definitions.

-We have found out that it is not always the case.

-That is why we want to know when exactly a symmetric simulation is a bisimulation.

Thank you for your attention! :-)