Coalgebraic Simulation.

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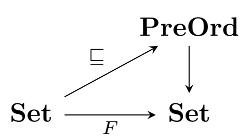
Abstract. Hello simulation!

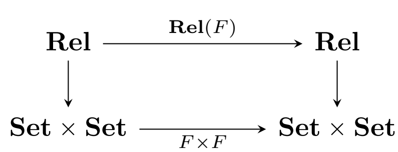
1 Coalgebraic Simulation

We show the category of preorders with monotone functions between them with **PreOrd**. In the diagrams, any arrow that shows a functor, but does not have a label is showing a forgetful functor. Also, we use **Rel** to refer to the category of binary relations. Assuming $R \in \mathbf{Obj}(\mathbf{Rel})$ and $R \subseteq X_1 \times X_2$, and $S \in \mathbf{Obj}(\mathbf{Rel})$ and $S \subseteq Y_1 \times Y_2$, then a morphism $f\colon R \to S$ in this category is the pair (f_1, f_2) of morhpisms in **Set**, where, $f_1\colon X_1 \to Y_1$ and $f_2\colon X_2 \to Y_2$, and for each $(x_1, x_2) \in R$ we have $(f_1(x_1), f_2(x_2)) \in S$. Also, we show projections of $R \in \mathbf{Obj}(\mathbf{Rel})$ with p_1 and p_2 that are morphisms in **Set**.

Definition 1.1 (A Preorder Over a Functor). Assuming $F\colon \mathbf{Set} \to \mathbf{Set}$ is a functor, we call $\sqsubseteq\colon \mathbf{Set} \to \mathbf{PreOrd}$ an order over the functor F iff the following diagram commutes:



Definition 1.2 (Relation Lifting). Assuming $F\colon \mathbf{Set} \to \mathbf{Set}$ is a functor, then we call $\mathbf{Rel}(F)\colon \mathbf{Rel} \to \mathbf{Rel}$ a relation lifting of F, where the following diagram commutes:



We take $\mathbf{Rel}(F)\colon \mathbf{Rel} \to \mathbf{Rel}$ to be the functor that for an arbitrary functor F takes a relation R, where $R \in \mathbf{Obj}(\mathbf{Rel})$ and $R \subseteq X_1 \times X_2$, and gives the relation that is the image of the function $\langle Fp_1, Fp_2\rangle\colon FR \to FX \times FY$.

Definition 1.3 (Bisimulation). For a functor $F\colon \mathbf{Set} \to \mathbf{Set}$, a bisimulation is a $\mathbf{Rel}(F)$-coalgebra in **Rel**.

Proposition 1.4. *Assuming that (R, α) is a $\mathbf{Rel}(F)$-coalgebra, where $\alpha = \beta_1 \times \beta_2$ in $\mathbf{Set} \times \mathbf{Set}$, then the following diagram commutes, and vice-versa:*

$$\begin{array}{ccccc} X_1 & \xleftarrow{p_1} & R & \xrightarrow{p_2} & X_2 \\ \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta_2} \\ FX_1 & \xleftarrow[Fp_1]{} & FR & \xrightarrow[Fp_2]{} & FX_2 \end{array}$$

We gave an introduction to Hughes and Jacobs paper. They also have a way to represent simulation relations. In the following, we try to find a suitable formalization for simulation relations, inspired by Hughes and Jacobs.

1.1 Relations as ~~Pullbacks~~ Spans(?)

We can not show every relation by pullbacks, but we can just show relations of the form

$$\{(a, b) \mid f(a) = g(b)\}$$

for some functions f and g, when we are in $\mathbf{Set}$, so we can not show every object in $\mathbf{Rel}$ using this approach, including $\mathbf{Rel}_{\sqsubseteq}(F)(R) \sqsubseteq_{X_2}; \mathbf{Rel}(F)(R); \sqsubseteq_{X_1}$ that is the target of simulation. Although we can show $\mathbf{Rel}_{\sqsubseteq}(F)(R) \sqsubseteq_{X_2}; \mathbf{Rel}(F)(R); \sqsubseteq_{X_1}$ as a span.

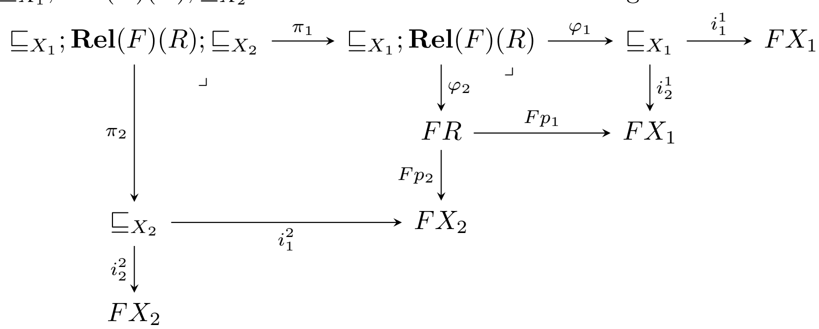
Assuming, we have a category $\mathbf{C}$, an object of the category of spans over $\mathbf{C}$ is (R, X_1, X_2, p_1, p_2) in the form of the following diagram:

$$\begin{array}{ccccc} & & R & & \\ & {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} & \\ X_1 & & & & X_2 \end{array}$$

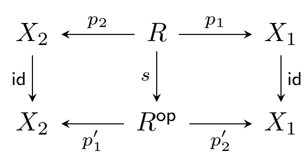
A morhpism from a span (R, X_1, X_2, p_1, p_2) to a span (S, Y_1, Y_2, q_1, q_2) is a morphism $f : R \to S$ in $\mathbf{C}$, for which exist $f_1 : X_1 \to Y_i$ and $f_2 : X_2 \to Y_j$, where $i, j \in \{1, 2\}$ and $i \neq j$, and they are in $\mathbf{C}$, that take part in the following commuting diagram:

$$\begin{array}{ccccc} X_2 & \xleftarrow{p_2} & R & \xrightarrow{p_1} & X_1 \\ \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_1} \\ Y_j & \xleftarrow[q_j]{} & S & \xrightarrow[q_i]{} & Y_i \end{array}$$

We define a F-simulation as the coalgebra of the object $\sqsubseteq_{X_1}; \mathbf{Rel}(F)(R); \sqsubseteq_{X_2}$ that has the following structure in $\mathbf{C}$:



We show that if we consider a relation R and its opposite are both simulation relations, then R is a bisimulation. To reach to that goal, we give a formal definition of what we mean by the opposite of R in our categorical setting that we show with R^{op}. $(R^{\mathsf{op}}, p'_1, p'_2)$ is a span, that is isomorphic to R via morphism $s\colon R \to R^{\mathsf{op}}$ in $\mathbf{Rel}$ that we call swap, and it commutes in the following commutative diagram:

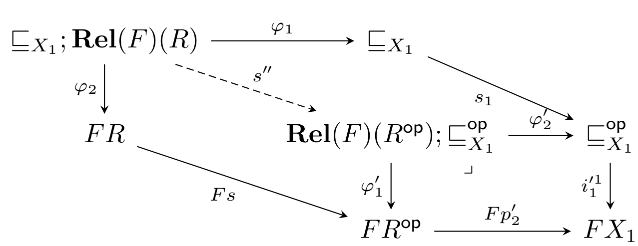


Lemma 1.5. *The relation* $(\sqsubseteq_{X_1}; \mathbf{Rel}(F)(R); \sqsubseteq_{X_2})^{\mathsf{op}}$ *is isomorphic to* $\sqsubseteq_{X_2^{\mathsf{op}}}$ $; \mathbf{Rel}(F)(R^{\mathsf{op}}); \sqsubseteq_{X_1^{\mathsf{op}}}$.

Proof. We set $s_1\colon \sqsubseteq_{X_1} \to \sqsubseteq_{X_1}^{\mathsf{op}}$ and $s_2\colon \sqsubseteq_{X_2} \to \sqsubseteq_{X_2}^{\mathsf{op}}$ to be the swaps of $\sqsubseteq_{X_1}$ and $\sqsubseteq_{X_2}$, respectively. Since we have

$$
\begin{aligned}
&i'^1_1 \cdot s_1 \cdot \varphi_1 \\
&\qquad = i_2^1 \cdot \varphi_2 \\
&\qquad = Fp_1 \cdot \varphi_2 \\
&\qquad = Fp'_2 \cdot Fs \cdot \varphi_2,
\end{aligned}
$$

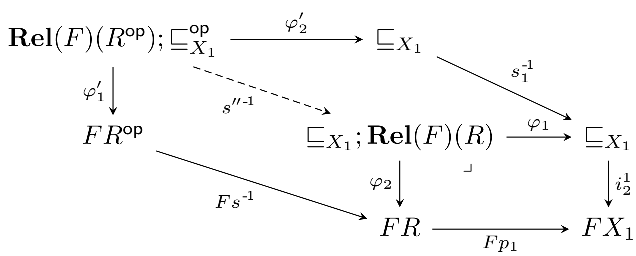
there exists the morphism $s''\colon \sqsubseteq_{X_1}; \mathbf{Rel}(F)(R) \to \mathbf{Rel}(F)(R^{\mathsf{op}}); \sqsubseteq_{X_1}^{\mathsf{op}}$ depicted in the following commutative diagram:



Similarily, we get $s''^{-1}\colon \sqsubseteq_{X_1};\mathbf{Rel}(F)(R) \to \mathbf{Rel}(F)(R^{\mathsf{op}});\sqsubseteq_{X_1}^{\mathsf{op}}$ since

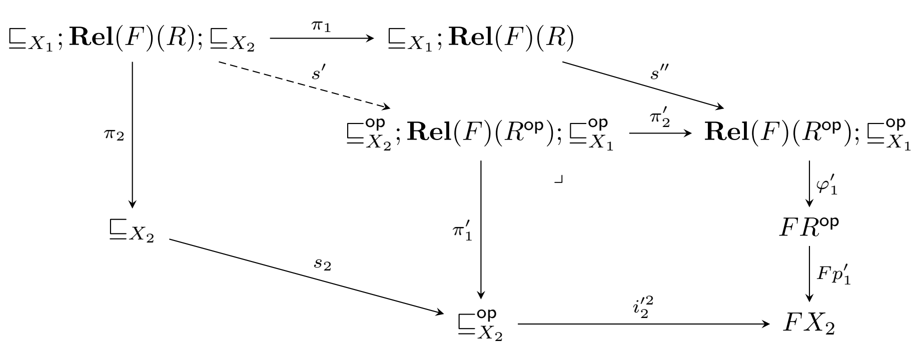
$$\begin{aligned} & i_2^1 \cdot s_1^{-1} \cdot \varphi_2' \\ &\quad = i_1'^1 \cdot \varphi_2' \\ &\quad = Fp_2' \cdot \varphi_1' \\ &\quad = Fp_1 \cdot Fs_1^{-1} \cdot \varphi_1', \end{aligned}$$

and it is depicted in the following diagram:

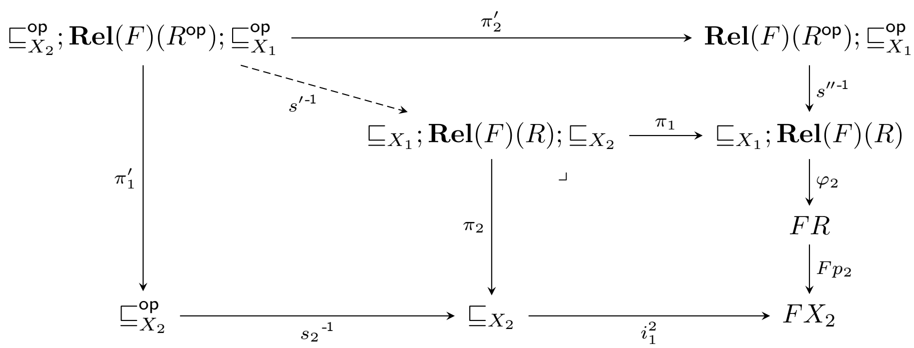


Obviously, s'' and s''^{-1} are each other's inverse, thus $\sqsubseteq_{X_1};\mathbf{Rel}(F)(R)$ and $\mathbf{Rel}(F)(R^{\mathsf{op}});\sqsubseteq_{X_1}^{\mathsf{op}}$ are isomorphic.

$$\begin{aligned} & Fp_1' \cdot \varphi_1' \cdot s'' \cdot \pi_1 \\ &\quad = Fp_1' \cdot Fs \cdot \varphi_2 \cdot \pi_1 \\ &\quad = Fp_2 \cdot \varphi_2 \cdot \pi_1 \\ &\quad = i_1^2 \cdot \pi_2 \\ &\quad = i_2'^2 \cdot s_2 \cdot \pi_2 \end{aligned}$$



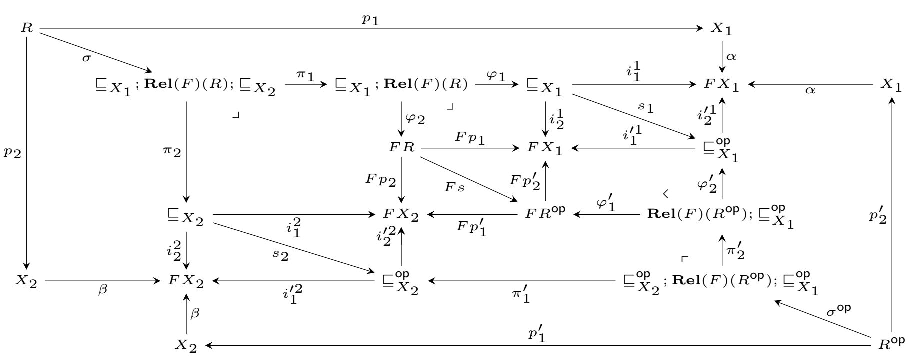
$$\begin{aligned} & Fp_2 \cdot \varphi_2 \cdot s''^{-1} \cdot \pi_2' \\ &\quad = Fp_2 \cdot Fs^{-1} \cdot \varphi_1' \cdot \pi_2' \\ &\quad = Fp_1' \cdot \varphi_1' \cdot \pi_2' \\ &\quad = i_2'^2 \cdot \pi_1' \\ &\quad = i_1^2 \cdot s_2^{-1} \cdot \pi_1' \end{aligned}$$



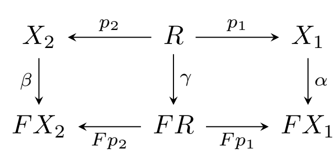
So, we could prove that $\sqsubseteq_{X_1}; \mathbf{Rel}(F)(R); \sqsubseteq_{X_2}$ and $\sqsubseteq^{op}_{X_2}; \mathbf{Rel}(F)(R^{op}); \sqsubseteq^{op}_{X_1}$ are isomorphic. $(\sqsubseteq_{X_1}; \mathbf{Rel}(F)(R); \sqsubseteq_{X_2})^{op}$ is isomorphic to $\sqsubseteq_{X_1}; \mathbf{Rel}(F)(R); \sqsubseteq_{X_2}$ by definition, so it is also isomorphic with $\sqsubseteq^{op}_{X_2}; \mathbf{Rel}(F)(R^{op}); \sqsubseteq^{op}_{X_1}$. □

Proposition 1.6. *Having* $\sigma \colon R \to \sqsubseteq_{X_2}; \mathbf{Rel}(F)(R); \sqsubseteq_{X_1}$ *and* $\sigma^{op} \colon R^{op} \to \sqsubseteq_{X_1}; \mathbf{Rel}(F)(R^{op}); \sqsubseteq_{X_2}$ *gives rise to a morphism* $\gamma \colon R \to \mathbf{Rel}(F)(R)$*, and vice-versa.*

Proof.

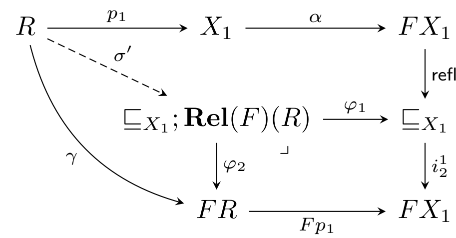


($\Leftarrow$): We assume that we have the morphism $\gamma \colon R \to FR$ such that the following diagram commutes:



(1)

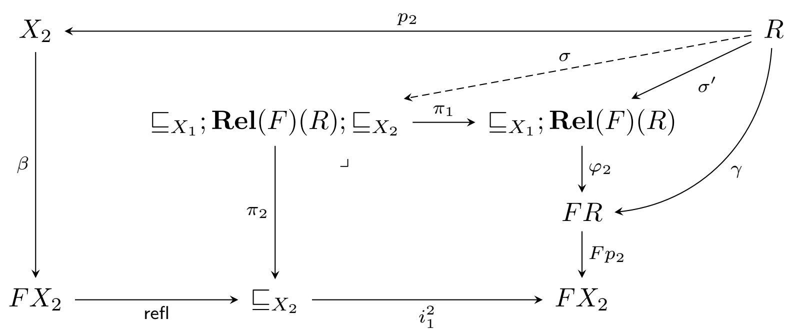
Since $\sqsubseteq_{X_1}$ and $\sqsubseteq_{X_1}$ preorders, they each have a morphism refl that pre-composed with their projections gives identity. As it is depicted in the following diagram the pullback property of $\sqsubseteq_{X_1}; \mathbf{Rel}(F)(R)$ gives us $\sigma' \colon R \to \sqsubseteq_{X_1}; \mathbf{Rel}(F)(R)$ in the following commutative diagram:



(2) {eq:diag-thm-sig'}

Then the pullback property of $\sqsubseteq_{X_1};\mathbf{Rel}(F)(R);\sqsubseteq_{X_2}$ gives us the existence of $\sigma\colon R \to \sqsubseteq_{X_1};\mathbf{Rel}(F)(R);\sqsubseteq_{X_2}$ in the following commutative diagram:

{eq:diag-thm-sig}



(3)

Now, we show that σ is a simulation:

$$
\begin{aligned}
&i_1^1 \cdot \varphi_1 \cdot \pi_1 \cdot \sigma \\
&\qquad = i_1^1 \cdot \varphi_1 \cdot \sigma' && /\!/ \ (3) \\
&\qquad = i_1^1 \cdot \mathsf{refl} \cdot \alpha \cdot p_1 && /\!/ \ (2) \\
&\qquad = \alpha \cdot p_1
\end{aligned}
$$

$$
\begin{aligned}
&i_2^2 \cdot \pi_2 \cdot \sigma \\
&\qquad = i_2^2 \cdot \mathsf{refl} \cdot \beta \cdot p_2 && /\!/ \ (3) \\
&\qquad = \beta \cdot p_2
\end{aligned}
$$

Considering that $s\colon R \to R^{\mathsf{op}}$ and $s'\colon \sqsubseteq_{X_1};\mathbf{Rel}(F)(R);\sqsubseteq_{X_2} \to \sqsubseteq_{X_2}^{\mathsf{op}};\mathbf{Rel}(F)(R^{\mathsf{op}});\sqsubseteq_{X_1}^{\mathsf{op}}$ are swapping isomorphisms, We set $\sigma^{\mathsf{op}}\colon R^{\mathsf{op}} \to \sqsubseteq_{X_2}^{\mathsf{op}};\mathbf{Rel}(F)(R^{\mathsf{op}});\sqsubseteq_{X_1}^{\mathsf{op}}$ to be $\sigma^{\mathsf{op}} = s' \cdot \sigma \cdot s^{\text{-}1}$. Now, we show that σ' is a simulation:

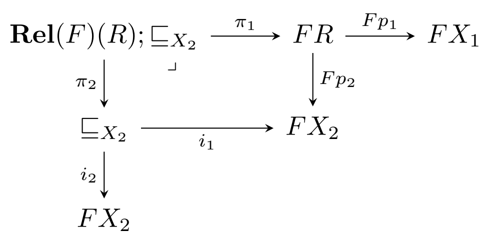
$$
\begin{aligned}
&i_2'^1 \cdot \varphi_2' \cdot \pi_2' \cdot \sigma^{\mathsf{op}} \\
&\qquad = i_2'^1 \cdot \varphi_2' \cdot \pi_2' \cdot s' \cdot \sigma \cdot s^{\text{-}1} \\
&\qquad = i_1^1 \cdot \varphi_1 \cdot \pi_1 \cdot \sigma \cdot s^{\text{-}1} \\
&\qquad = \alpha \cdot p_1 \cdot s^{\text{-}1}
\end{aligned}
$$

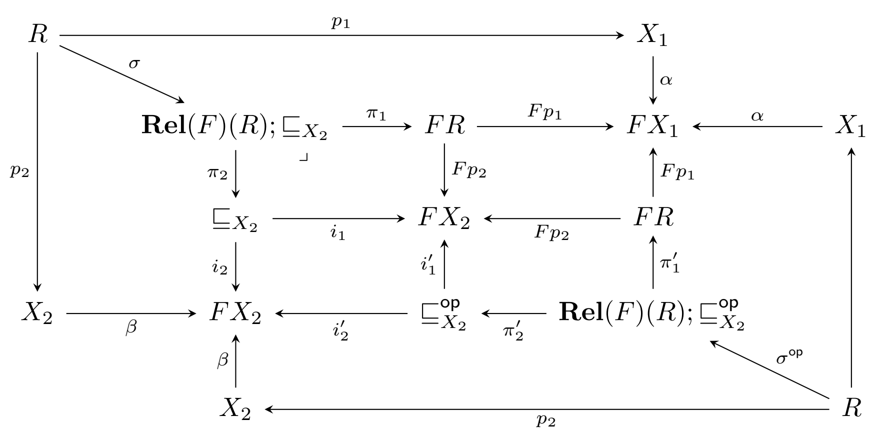
$$= \alpha \cdot p_2'$$

$$i_1'^2 \cdot \pi_1' \cdot \sigma^{\mathsf{op}} =$$
$$= i_1'^2 \cdot \pi_1' \cdot s' \cdot \sigma \cdot s^{-1}$$
$$= i_2^2 \cdot \pi_2 \cdot \sigma \cdot s^{-1}$$
$$= \beta \cdot p_2 \cdot s^{-1}$$
$$= \beta \cdot p_1'$$

1.2 Simulation with one relation composition

We recall everything we had in the previous section. Although we want to work with the functor that takes $R \subseteq X_1 \times X_2$ and gives $\mathbf{Rel}(F)(R); \sqsubseteq_{X_2}$.





Proposition 1.7. *Assuming $R \subseteq X \times X$, then if we have $\sigma\colon R \to \mathbf{Rel}(F)(R); \sqsubseteq_X$ as a simulation for R, and R is reflexive, then we have $\gamma\colon R \to \mathbf{Rel}(F)(R)$ as a bisimulation for R, and vice-versa.*

Proof. $(\Rightarrow)$:

$$Fp_2 \cdot \pi_1 \cdot \sigma$$
$$= i_1 \cdot \pi_2 \cdot \sigma$$
$$= i_2' \cdot s \cdot \pi_2 \cdot \sigma$$
$$=$$

1.3 Using Lax Pullbacks (Comma Objects) to Model Simulation

A big concern with this approach is that Comma Objects are defined in a 2-category, so we can not define them in **Set**, while our main inspirational example is coming from **Set**.

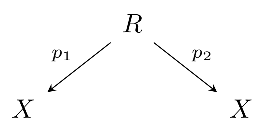
1.4 Working in Set First, Like Hughes and Jacobs

1.5 Choosing a suitable order for our setting

Maybe we can first choose a suitable order on $T(\Sigma_{\mathsf{v}}\mu\Sigma \times D(\mu\Sigma, \mu\Sigma))$ and then prove that if a relation and its inverse is a simulation then it is a bisimulation as well. Maybe T being ω-continuous can give the ordering. It can be something easier that relates to termination as well! That if a term has a big-step evaluation, then it is bigger than or equal to any other term, and if it does not, then it is less than or equal to any other term.

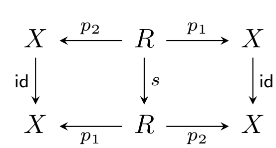
2 Symmetric Simulation is Bisimulation

Definition 2.1 (Graph). In a category $\mathbf{C}$ a graph is a tuple (R, X) of the following form:



Graphs over $\mathbf{C}$ form a category that we show by $\mathbf{Gra}(\mathbf{C})$.

Definition 2.2 (Symmetric Graph). A graph (R, X) is symmetric iff there exists an endomorphism $s\colon R \to R$, such that the following diagram commutes



and $s \cdot s = \mathsf{id}$. We call s a *swap* for R.

{lem:gra-sym}

Lemma 2.3. *Symmetry of a graphs over preserved a functor.*

Definition 2.4 (Relation). A relation in a category $\mathbf{C}$ is a graph (R, X) where $\langle p_1, p_2\rangle\colon R \to X \times Y$ is monic. Relations over $\mathbf{C}$ form a category that we show by $\mathbf{Rel}(\mathbf{C})$.

Definition 2.5 (Jointly Monic). A pair of morphisms $p_1, p_2\colon R \to X$ is jointly monic iff for every pair of morphisms $f, g\colon A \to R$ assuming that $p_1 \cdot f = p_1 \cdot g$ and $p_2 \cdot f = p_2 \cdot g$ then $f = g$.

{prop:rel-joi-mon}

Proposition 2.6. *A graph (R, X) is a relation iff p_1 and p_2 are jointly monic.*

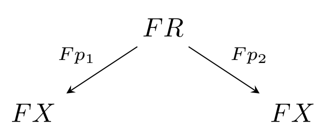
Proof. ($\Rightarrow$): We assume that for morphisms $f, g\colon A \to R$ we have $p_1 \cdot f = p_1 \cdot g$ and $p_2 \cdot f = p_2 \cdot g$, and we want to prove that $f = g$. Assuming that $\pi_1, \pi_2\colon X \times X \to X$ are projections of $X \times X$, then we have:

$$\begin{aligned}&\langle p_1, p_2\rangle \cdot f\\ &\quad = \langle p_1 \cdot f, p_2 \cdot f\rangle\\ &\quad = \langle p_1 \cdot g, p_2 \cdot g\rangle\\ &\quad = \langle p_1, p_2\rangle \cdot g\end{aligned}$$

Since $\langle p_1, p_2\rangle$ is moinc, from $\langle p_1, p_2\rangle \cdot f = \langle p_1, p_2\rangle \cdot g$ we get $f = g$.

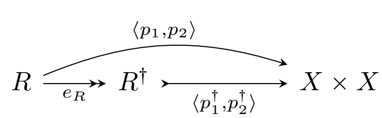
($\Leftarrow$): Assuming for some morphisms $f, g\colon A \to R$ we have $\langle p_1, p_2\rangle \cdot f = \langle p_1, p_2\rangle \cdot g$ we need to prove $f = g$. From $\langle p_1, p_2\rangle \cdot f = \langle p_1, p_2\rangle \cdot g$ we get $\langle p_1 \cdot f, p_2 \cdot f\rangle = \langle p_1 \cdot g, p_2 \cdot g\rangle$. Assuming that $\pi_1, \pi_2\colon X \times X \to X$ are projections of $X \times X$, then we have $\pi_1 \cdot \langle p_1 \cdot f, p_2 \cdot f\rangle = \pi_1 \cdot \langle p_1 \cdot g, p_2 \cdot g\rangle$, and then $p_1 \cdot f = p_1 \cdot g$. Similarly we also get $p_2 \cdot f = p_2 \cdot g$. So, since p_1 and p_2 are jointly monic, then we have $f = g$. □

We need to work with endofunctors over $\mathbf{C}$ that are lifted over $\mathbf{Rel}(\mathbf{C})$, for which we need to first define endofunctors lifted over $\mathbf{Gra}(\mathbf{C})$. Lifting from $\mathbf{C}$ to $\mathbf{Gra}(\mathbf{C})$ is easy. For $F\colon \mathbf{C} \to \mathbf{C}$ we define $F_{\mathbf{Gra}}\colon \mathbf{Gra}(\mathbf{C}) \to \mathbf{Gra}(\mathbf{C})$ as a functor that takes a graph (R, X), and gives (FR, FX), were F is also applied on legs of the graph, i.e., $p_1, p_2\colon R \to X$, so, we get the following graph:



This lifting does not work for **Rel**. As an example, if we set F to be the powerset functor $\mathcal{P}$, then $(\mathcal{P}R, \mathcal{P}X)$ is not necessarily a relation anymore. For example, if we take $R = \{(1,0),(0,1),(0,0),(1,1)\}$, then taking $\{\{(1,0),(0,1),(0,0),(1,1)\}\}$ and $\{\{(1,0),(0,1),(0,0)\}\}$ as elements of $\mathcal{P}R$, the morphism $\langle \mathcal{P}p_1, \mathcal{P}p_2\rangle$ maps them both to $(\{0,1\},\{0,1\})$ so it is not monic.

To cope with this, we assume the following epi-mono decomposition for $(R, X) \in \mathbf{Rel}(\mathbf{C})$:



We can define $(-)^\dagger$ as a functor from $\mathbf{Gra}(\mathbf{C}) \to \mathbf{Rel}(\mathbf{C})$, then we define $F_{\mathbf{Rel}} \colon \mathbf{Rel}(\mathbf{C}) \to \mathbf{Rel}(\mathbf{C})$ to take (R, X) to the following relation:

$$\begin{array}{ccccc} & & (FR)^\dagger & & \\ & \swarrow^{(Fp_1)^\dagger} & \Big\downarrow{\scriptstyle \langle (Fp_1)^\dagger, (Fp_2)^\dagger \rangle} & \searrow^{(Fp_2)^\dagger} & \\ FX & & & & FX \\ & & FX \times FX & & \end{array}$$

{lem:norm-simp}

Lemma 2.7. *Assuming that we have the following commutative diagram:*

$$\begin{array}{ccccc} X & \xleftarrow{p_1} & R & \xrightarrow{p_2} & X \\ \alpha\downarrow & & \downarrow\sigma & & \downarrow\alpha \\ FX & \xleftarrow[Fp_1]{} & FR & \xrightarrow[Fp_2]{} & FX \end{array}$$

Then there exists $\sigma^\dagger \colon R \to (FR)^\dagger$ in the following diagram that is also commutative:

$$\begin{array}{ccccc} X & \xleftarrow{p_1} & R & \xrightarrow{p_2} & X \\ \alpha\downarrow & & \downarrow\sigma^\dagger & & \downarrow\alpha \\ FX & \xleftarrow[Fp_1^\dagger]{} & (FR)^\dagger & \xrightarrow[Fp_2^\dagger]{} & FX \end{array}$$

Proof. The proof is trivial considering that $\sigma^\dagger = e_{FR} \cdot \sigma$, where e_{FR} is the epimorphism in the epi-mono factorization of $\langle Fp_1, Fp_2 \rangle$, as depicted in the following diagram:

$$\begin{array}{ccccc} X & \xleftarrow{p_1} & R & \xrightarrow{p_2} & X \\ \alpha\downarrow & & \downarrow\sigma & & \downarrow\alpha \\ FX & \xleftarrow[Fp_1]{} & FR & \xrightarrow[Fp_2]{} & FX \\ \mathsf{id}\downarrow & & \downarrow e_{FR} & & \downarrow\mathsf{id} \\ FX & \xleftarrow[Fp_1^\dagger]{} & (FR)^\dagger & \xrightarrow[Fp_2^\dagger]{} & FX \end{array}$$

□

We show this relation with $F_{\mathbf{Rel}}(R, X)$.

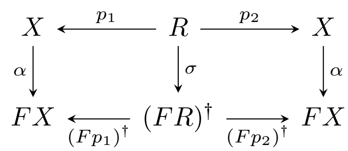
{def:sim} **Definition 2.8 (Simulation).** A coalgebra $\sigma \colon R \to (FR)^\dagger$ is a simulation over the F-coalgebra $\alpha \colon X \to FX$ iff the following diagram is lax-commutative:

{eq:diag-lax-sim}

$$\begin{array}{ccccc} X & \xleftarrow{p_1} & R & \xrightarrow{p_2} & X \\ \alpha\downarrow & \leqslant & \downarrow\sigma & \leqslant & \downarrow\alpha \\ FX & \xleftarrow[(Fp_1)^\dagger]{} & (FR)^\dagger & \xrightarrow[(Fp_2)^\dagger]{} & FX \end{array} \qquad (4)$$

{def:bisim} **Definition 2.9 (Bisimulation).** The morphism σ in Definition 2.8 is a bisimulation iff the mentioned diagram is fully commutative.

Remark 2.10. The mentioned definition of bisimulation is actually, the classical one in the literature that is to have the following commutative diagram:



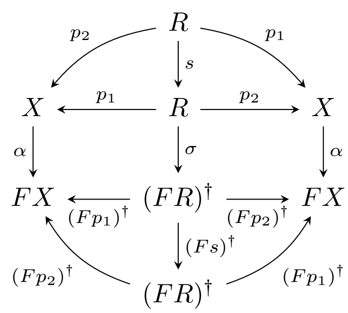
It may look different because we have FX and not $(FX)^\dagger$, but they are the same. An object $X \in \mathbf{Obj}(\mathbf{C})$ is $(X, X) \in \mathbf{Rel}(\mathbf{C})$ having id as its legs. Meaning that the $(FX)^\dagger = FX$.

{prop:iff-sim-bsim}

Proposition 2.11. *Assuming that we have a bisimulation σ for R, we have the following equation:*

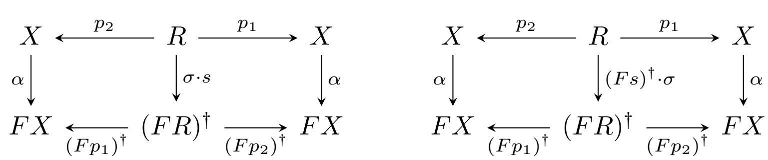
$$\sigma \cdot s = (Fs)^\dagger \cdot \sigma$$

Proof. We recall that by Lemma 2.3, $F_{\mathbf{Rel}}(R, X)$ is symmetric with the swap $(Fs)^\dagger$. Assuming that σ is a bisimulation, we have the following commutative diagram:



(5) {eq:diag-sym-rel}

And it entails that the following diagrams are also commutative:



So, since $(Fp_1)^\dagger$ and $(Fp_2)^\dagger$ are jointly monic (because $F_{\mathbf{Rel}}(R, X)$ is a relation and Proposition 2.6) we have $\sigma \cdot s = (Fs)^\dagger \cdot \sigma$. ◻

Corollary 2.12. *Assuming σ_1 and σ_2 are simulations of type $R \to (FR)^\dagger$, and R is symmetric and both σ_1 and σ_2 satisfy the following property:*

$$(Fs)^\dagger \cdot \sigma = \sigma \cdot s$$

Then $\sigma_1 = \sigma_2$.

Proof. As the mentioned property is equivalent with σ being a bisimulation, and bisimulation is unique, then $\sigma_1 = \sigma_2$.

Now, we give a counter example of a symmetric relation on **Set** that is a simulation according to Definition 2.8, i.e, exists the morphism σ that commutes laxly in (4), but σ is not a coalgebraic bisimulation, although the relation that we give is clearly a bisimulation in the classic sense. We set $R = \{(A, B), (B, A), (C_1, C_2), (C_2, C_1), (C_2', C_2), (C_2, C_2'), (C_2, C_2)\}$, $F = \mathbf{Id}$, $\sqsubseteq = \Delta \cup \{(C_1, C_2), (C_2, C_2')\}$, and the coalgebra α is defined with the following set of reductions:

$$A \to C_1 \qquad B \to C_2 \qquad C_1 \to C_1 \qquad C_2 \to C_2 \qquad C_2' \to C_2$$

And finally, we define σ as follows:

$$\sigma(w) = \begin{cases} (\alpha \cdot p_1(w), \alpha \cdot p_2(w)) & w \neq (B, A) \\ (C_2', C_2) & w = (B, A) \end{cases}$$

It is easy to check that the conditions $\alpha \cdot p_1 \sqsubseteq (Fp_1)^\dagger \cdot \sigma$ and $(Fp_2)^\dagger \cdot \sigma \sqsubseteq \alpha \cdot p_2$ are satisfied. For every $w \in R$ if $w \neq (B, A)$ then for $i \in \{1, 2\}$, we have $\alpha \cdot p_i = (Fp_i)^\dagger \cdot \sigma$, and for $w = (B, A)$ we have $\alpha \cdot p_1(B, A) = C_2 \sqsubseteq C_2' = (Fp_1)^\dagger \cdot \sigma(B, A)$, and $\alpha \cdot p_2(B, A) = C_1 \sqsubseteq C_2 = (Fp_2)^\dagger \cdot \sigma(B, A)$. And σ is not a coalgebraic bisimulation as $\alpha \cdot p_1(B, A) = C_2 \neq C_2' = (Fp_1)^\dagger \cdot \sigma(B, A)$.

An interesting question would be to find out what conditions σ should have (maybe we have the answer to this! Proposition 2.11), or how it should be constructed (perhaps based on a given poset) so that it will also be a coalgebraic bisimulation if R is symmetric. Another avenue would be to give another definition for simulation that does not have this issue.

Well! This counter example does not work! Because the described order $\sqsubseteq$ does not satisfy the condition mentioned in Jacobs's paper. The condition is that the order on FX should satisfy the property that for a morphism $f\colon X \to Y$ the morphism $Ff\colon FX \to FY$ preserves $\sqsubseteq$. Probably, the only poset that has this property for **Id** is Δ. If there is a counter-example, it is true for another functor.

Example 2.13. Another counter-example! Assume that $F = \mathcal{P}$, and take $R = \{(1, 2), (2, 1), (1, 3), (3, 1)\}$, and $X = \{1, 2, 3\}$. $\alpha(x) = X$ for every $x \in X$, and σ is defined as below:

$$\sigma(w) = \begin{cases} R & w \neq (1, 3) \\ R \backslash \{(1, 2)\} & w = (1, 3) \end{cases}$$

In this scenario, σ is a simulation, but it is not a bisimulation. It is a simulation since for every $w \in R$ we have $(\mathcal{P}p_1(\sigma(w))) = X$, thus for every $x \in X$, $\alpha(x) = X \subseteq X$. Also, $\mathcal{P}p_2(\sigma(w)) \subseteq \alpha(p_2(w))$ as $\alpha(p_2(w)) = X$ for every $w \in R$. But it is not a bisimulation, since $\alpha \cdot p_2(1, 3) = \alpha(3) = X \neq (\mathcal{P}p_2)^\dagger(\sigma(1, 3)) = \{1, 3\}$.

Example 2.14. And another counter-example!!! Assume that $F = \mathcal{P}$, and take $R = X \times X \backslash \{(1,3),(3,1)\}$, and $X = \{1,2,3\}$. α is defined as below:

$$\alpha(x) = \begin{cases} \{1,2\} & x = 1 \\ \{2,3\} & x = 2 \\ \{3\} & x = 3 \end{cases}$$

And σ_1 is defined as below:

$$\sigma_1(w) = \begin{cases} \{(1,2),(2,2)\} & w = (1,2) \\ \{(2,1),(3,2)\} & w = (2,1) \\ \{(1,2),(2,1)\} & w = (1,1) \\ \{(2,2),(3,3)\} & w \in \{(2,2),(3,2)\} \\ \{(2,3),(3,3)\} & w = (2,3) \\ \{(3,3)\} & w = (3,3) \end{cases}$$

$$(\mathcal{P}p_1)^\dagger \cdot \sigma_1(w) = \begin{cases} \{1,2\} & w = (1,2) \\ \{2,3\} & w = (2,1) \\ \{1,2\} & w = (1,1) \\ \{2,3\} & w \in \{(2,2),(3,2)\} \\ \{2,3\} & w = (2,3) \\ \{3\} & w = (3,3) \end{cases} (\mathcal{P}p_2)^\dagger \cdot \sigma_1(w) = \begin{cases} \{2\} & w = (1,2) \\ \{1,2\} & w = (2,1) \\ \{1,2\} & w = (1,1) \\ \{2,3\} & w \in \{(2,2),(3,2)\} \\ \{3\} & w = (2,3) \\ \{3\} & w = (3,3) \end{cases} \blacksquare$$

$$\sigma_1'(w) = \begin{cases} \{(1,2),(2,2),(2,3)\} & w = (1,2) \\ \{(2,1),(2,2),(3,2)\} & w = (2,1) \\ \{(1,2),(2,1)\} & w = (1,1) \\ \{(2,2),(3,3)\} & w = (2,2) \\ \{(2,2),(3,3),(3,2)\} & w = (3,2) \\ \{(2,3),(2,2),(3,3)\} & w = (2,3) \\ \{(3,3)\} & w = (3,3) \end{cases}$$

$$(\mathcal{P}p_1)^\dagger \cdot \sigma_1'(w) = \begin{cases} \{1,2\} & w = (1,2) \\ \{2,3\} & w = (2,1) \\ \{1,2\} & w = (1,1) \\ \{2,3\} & w = (2,2) \\ \{2,3\} & w = (3,2) \\ \{2,3\} & w = (2,3) \\ \{3\} & w = (3,3) \end{cases} (\mathcal{P}p_2)^\dagger \cdot \sigma_1'(w) = \begin{cases} \{2,3\} & w = (1,2) \\ \{1,2\} & w = (2,1) \\ \{1,2\} & w = (1,1) \\ \{2,3\} & w = (2,2) \\ \{2,3\} & w = (3,2) \\ \{2,3\} & w = (2,3) \\ \{3\} & w = (3,3) \end{cases}$$

σ_1' is not a simulation!

$$\sigma_1''(w) = \begin{cases} \{(1,2)\} & w = (1,2) \\ \{(2,1)\} & w = (2,1) \\ \{(1,2),(2,1)\} & w = (1,1) \\ \{(2,2),(3,3)\} & w = (2,2) \\ \{(3,3)\} & w = (3,2) \\ \{(3,3)\} & w = (2,3) \\ \{(3,3)\} & w = (3,3) \end{cases}$$

$$\begin{aligned} \sigma_1 &\sqsubseteq \sigma_1' \\ \sigma_3 &\sqsubseteq \sigma_1' \\ \beta &= \sigma_1' \end{aligned}$$

$$(\mathcal{P}s)^\dagger \cdot \sigma_1 \cdot s(w) = \begin{cases} \{(1,2),(2,3)\} & w = (1,2) \\ \{(2,1),(2,2)\} & w = (2,1) \\ \{(1,2),(2,1)\} & w = (1,1) \\ \{(2,2),(3,3)\} & w = (2,2) \\ \{(3,2),(3,3)\} & w = (3,2) \\ \{(2,2),(3,3)\} & w = (2,3) \\ \{(3,3)\} & w = (3,3) \end{cases}$$

$$(\mathcal{P}p_1)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma_1 \cdot s(w) = \begin{cases} \{1,2\} & w = (1,2) \\ \{2\} & w = (2,1) \\ \{1,2\} & w = (1,1) \\ \{2,3\} & w = (2,2) \\ \{3\} & w = (3,2) \\ \{2,3\} & w = (2,3) \\ \{3\} & w = (3,3) \end{cases} (\mathcal{P}p_2)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma_1 \cdot s(w) = \begin{cases} \{2,3\} & w = (1,2) \\ \{1,2\} & w = (2,1) \\ \{1,2\} & w = (1,1) \\ \{2,3\} & w = (2,2) \\ \{2,3\} & w = (3,2) \\ \{2,3\} & w = (2,3) \\ \{3\} & w = (3,3) \end{cases}$$

In this scenario, σ_1 is a simulation, but it is not a bisimulation. σ_1', σ_1'' and $(\mathcal{P}s)^\dagger \cdot \sigma_1 \cdot s$ are neither. We can not make σ_1 bigger here to make it a bisimulation as $\alpha \cdot p_1(3,2) = \{3\} \subsetneq \{2,3\} = (\mathcal{P}p_1)^\dagger \cdot \sigma_1(3,2)$.

The following is also a simulation and not a bisimulation:

$$\sigma_2(w) = \begin{cases} \{(1,2),(2,2)\} & w = (1,2) \\ \{(2,1),(3,2)\} & w = (2,1) \\ \{(1,2),(2,1)\} & w = (1,1) \\ \{(2,2),(3,3)\} & w = (2,2) \\ \{(2,3),(3,3)\} & w = (2,3) \\ \{(3,3)\} & w \in \{(3,2),(3,3)\} \end{cases}$$

$$(\mathcal{P}p_1)^\dagger \cdot \sigma_2(w) = \begin{cases} \{1,2\} & w=(1,2) \\ \{2,3\} & w=(2,1) \\ \{1,2\} & w=(1,1) \\ \{2,3\} & w=(2,2) \\ \{2,3\} & w=(2,3) \\ \{3\} & w \in \{(3,2),(3,3)\} \end{cases}$$

$$(\mathcal{P}p_2)^\dagger \cdot \sigma_2(w) = \begin{cases} \{2\} & w=(1,2) \\ \{1,2\} & w=(2,1) \\ \{1,2\} & w=(1,1) \\ \{2,3\} & w=(2,2) \\ \{3\} & w=(2,3) \\ \{3\} & w \in \{(3,2),(3,3) \end{cases}$$

$$\sigma_2'(w) = \begin{cases} \{(1,2),(2,2),(2,3)\} & w=(1,2) \\ \{(2,1),(2,2),(3,2)\} & w=(2,1) \\ \{(1,2),(2,1)\} & w=(1,1) \\ \{(2,2),(3,3)\} & w=(2,2) \\ \{(3,3),(3,2)\} & w=(3,2) \\ \{(2,3),(3,3)\} & w=(2,3) \\ \{(3,3)\} & w=(3,3) \end{cases}$$

$$(\mathcal{P}p_1)^\dagger \cdot \sigma_2'(w) = \begin{cases} \{1,2\} & w=(1,2) \\ \{2,3\} & w=(2,1) \\ \{1,2\} & w=(1,1) \\ \{2,3\} & w=(2,2) \\ \{3\} & w=(3,2) \\ \{2,3\} & w=(2,3) \\ \{3\} & w=(3,3) \end{cases}$$

$$(\mathcal{P}p_2)^\dagger \cdot \sigma_2'(w) = \begin{cases} \{2,3\} & w=(1,2) \\ \{1,2\} & w=(2,1) \\ \{1,2\} & w=(1,1) \\ \{2,3\} & w=(2,2) \\ \{2,3\} & w=(3,2) \\ \{3\} & w=(2,3) \\ \{3\} & w=(3,3) \end{cases}$$

$$(\mathcal{P}s)^\dagger \cdot \sigma_2 \cdot s(w) = \begin{cases} \{(1,2),(2,3)\} & w=(1,2) \\ \{(2,1),(2,2)\} & w=(2,1) \\ \{(1,2),(2,1)\} & w=(1,1) \\ \{(2,2),(3,3)\} & w=(2,2) \\ \{(3,3),(3,2)\} & w=(3,2) \\ \{(3,3)\} & w=(2,3) \\ \{(3,3)\} & w=(3,3) \end{cases}$$

$$(\mathcal{P}p_1)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma_2 \cdot s(w) = \begin{cases} \{1,2\} & w=(1,2) \\ \{2\} & w=(2,1) \\ \{1,2\} & w=(1,1) \\ \{2,3\} & w=(2,2) \\ \{3\} & w=(3,2) \\ \{2\} & w=(2,3) \\ \{3\} & w=(3,3) \end{cases}$$

$$(\mathcal{P}p_2)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma_2 \cdot s(w) = \begin{cases} \{2,3\} & w=(1,2) \\ \{1,2\} & w=(2,1) \\ \{1,2\} & w=(1,1) \\ \{2,3\} & w=(2,2) \\ \{2,3\} & w=(3,2) \\ \{3\} & w=(2,3) \\ \{3\} & w=(3,3) \end{cases}$$

σ_2 is a simulation, σ_2' is a bisimulation, and $(\mathcal{P}s)^\dagger \cdot \sigma_2 \cdot s$ is neither. The following is both a simulation and a bisimulation:

$$\sigma_3(w) = \begin{cases} \{(1,2),(2,2),(2,3)\} & w = (1,2) \\ \{(2,1),(2,2),(3,2)\} & w = (2,1) \\ \{(1,2),(2,1)\} & w = (1,1) \\ \{(2,2),(3,3)\} & w = (2,2) \\ \{(2,3),(3,3)\} & w = (2,3) \\ \{(3,2),(3,3)\} & w = (3,2) \\ \{(3,3)\} & w = (3,3) \end{cases}$$

$$(\mathcal{P}p_1)^\dagger \cdot \sigma_3(w) = \begin{cases} \{1,2\} & w = (1,2) \\ \{2,3\} & w = (2,1) \\ \{1,2\} & w = (1,1) \\ \{2,3\} & w = (2,2) \\ \{2,3\} & w = (2,3) \\ \{3\} & w = (3,2) \\ \{3\} & w = (3,3) \end{cases} (\mathcal{P}p_2)^\dagger \cdot \sigma_3(w) = \begin{cases} \{2,3\} & w = (1,2) \\ \{1,2\} & w = (2,1) \\ \{1,2\} & w = (1,1) \\ \{2,3\} & w = (2,2) \\ \{3\} & w = (2,3) \\ \{2,3\} & w = (3,2) \\ \{3\} & w = (3,3) \end{cases}$$

The following is also a simulation and not a bisimulation:

$$\sigma_4(w) = \begin{cases} \{(1,2),(2,2),(3,2),(2,3),(3,3)\} & w = (1,2) \\ \{(1,1),(2,1),(1,2),(2,2),(3,2)\} & w = (2,1) \\ \{(1,2),(2,1)\} & w = (1,1) \\ \{(2,2),(3,3)\} & w = (2,2) \\ \{(2,3),(3,3)\} & w = (2,3) \\ \{(1,2),(2,2),(3,2),(2,3),(3,3)\} & w = (3,2) \\ \{(3,3)\} & w = (3,3) \end{cases}$$

$$(\mathcal{P}p_1)^\dagger \cdot \sigma_4(w) = \begin{cases} \{1,2,3\} & w = (1,2) \\ \{1,2,3\} & w = (2,1) \\ \{1,2\} & w = (1,1) \\ \{2,3\} & w = (2,2) \\ \{2,3\} & w = (2,3) \\ \{1,2,3\} & w = (3,2) \\ \{3\} & w = (3,3) \end{cases} (\mathcal{P}p_2)^\dagger \cdot \sigma_4(w) = \begin{cases} \{2,3\} & w = (1,2) \\ \{1,2\} & w = (2,1) \\ \{1,2\} & w = (1,1) \\ \{2,3\} & w = (2,2) \\ \{3\} & w = (2,3) \\ \{2,3\} & w = (3,2) \\ \{3\} & w = (3,3) \end{cases}$$

$$\sigma_4''(w) = \begin{cases} \{(1,2),(2,2),(2,3)\} & w = (1,2) \\ \{(2,1),(2,2),(3,2)\} & w = (2,1) \\ \{(1,2),(2,1)\} & w = (1,1) \\ \{(2,2),(3,3)\} & w = (2,2) \\ \{(2,3),(3,3)\} & w = (2,3) \\ \{(3,2),(3,3)\} & w = (3,2) \\ \{(3,3)\} & w = (3,3) \end{cases}$$

$$(\mathcal{P}p_1)^\dagger \cdot \sigma_4''(w) = \begin{cases} \{1,2\} & w = (1,2) \\ \{2,3\} & w = (2,1) \\ \{1,2\} & w = (1,1) \\ \{2,3\} & w = (2,2) \\ \{2,3\} & w = (2,3) \\ \{3\} & w = (3,2) \\ \{3\} & w = (3,3) \end{cases} \quad (\mathcal{P}p_2)^\dagger \cdot \sigma_4''(w) = \begin{cases} \{2,3\} & w = (1,2) \\ \{1,2\} & w = (2,1) \\ \{1,2\} & w = (1,1) \\ \{2,3\} & w = (2,2) \\ \{3\} & w = (2,3) \\ \{2,3\} & w = (3,2) \\ \{3\} & w = (3,3) \end{cases}$$

$$(\mathcal{P}s)^\dagger \cdot \sigma_4 \cdot s(w) = \begin{cases} \{(1,1),(1,2),(2,1),(2,2),(2,3)\} & w = (1,2) \\ \{(2,1),(2,2),(2,3),(3,2),(3,3)\} & w = (2,1) \\ \{(1,2),(2,1)\} & w = (1,1) \\ \{(2,2),(3,3)\} & w = (2,2) \\ \{(2,1),(2,2),(2,3),(3,2),(3,3)\} & w = (2,3) \\ \{(3,2),(3,3)\} & w = (3,2) \\ \{(3,3)\} & w = (3,3) \end{cases}$$

$$(\mathcal{P}p_1)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma_4 \cdot s(w) = \begin{cases} \{1,2\} & w = (1,2) \\ \{2,3\} & w = (2,1) \\ \{1,2\} & w = (1,1) \\ \{2,3\} & w = (2,2) \\ \{2,3\} & w = (2,3) \\ \{3\} & w = (3,2) \\ \{3\} & w = (3,3) \end{cases} \quad (\mathcal{P}p_2)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma_4 \cdot s(w) = \begin{cases} \{1,2,3\} & w = (1,2) \\ \{1,2,3\} & w = (2,1) \\ \{1,2\} & w = (1,1) \\ \{2,3\} & w = (2,2) \\ \{1,2,3\} & w = (2,3) \\ \{2,3\} & w = (3,2) \\ \{3\} & w = (3,3) \end{cases} \blacksquare$$

σ_4 is a simulation, σ_4'' is a bisimulation, and $(\mathcal{P}s)^\dagger \cdot \sigma_4 \cdot s$ is neither.

The following is also a simulation and not a bisimulation:

$$\sigma_5(w) = \begin{cases} \{(1,2),(2,2)\} & w=(1,2) \\ \{(2,2),(3,2)\} & w=(2,1) \\ \{(1,1),(2,1)\} & w=(1,1) \\ \{(2,2),(3,2)\} & w=(2,2) \\ \{(2,3),(3,3)\} & w=(2,3) \\ \{(3,3)\} & w=(3,2) \\ \{(3,3)\} & w=(3,3) \end{cases}$$

$$(\mathcal{P}p_1)^\dagger \cdot \sigma_5(w) = \begin{cases} \{1,2\} & w=(1,2) \\ \{2,3\} & w=(2,1) \\ \{1,2\} & w=(1,1) \\ \{2,3\} & w=(2,2) \\ \{2,3\} & w=(2,3) \\ \{3\} & w=(3,2) \\ \{3\} & w=(3,3) \end{cases} (\mathcal{P}p_2)^\dagger \cdot \sigma_5(w) = \begin{cases} \{2\} & w=(1,2) \\ \{2\} & w=(2,1) \\ \{1\} & w=(1,1) \\ \{2\} & w=(2,2) \\ \{3\} & w=(2,3) \\ \{3\} & w=(3,2) \\ \{3\} & w=(3,3) \end{cases}$$

The following is also a simulation and not a bisimulation:

$$\sigma_6(w) = \begin{cases} \{(1,2),(2,2)\} & w=(1,2) \\ \{(2,1),(3,1)\} & w=(2,1) \\ \{(1,2),(2,1)\} & w=(1,1) \\ \{(2,3),(3,3)\} & w=(2,2) \\ \{(2,3),(3,3)\} & w=(2,3) \\ \{(3,2)\} & w=(3,2) \\ \{(3,3)\} & w=(3,3) \end{cases}$$

$$(\mathcal{P}p_1)^\dagger \cdot \sigma_6(w) = \begin{cases} \{1,2\} & w=(1,2) \\ \{2,3\} & w=(2,1) \\ \{1,2\} & w=(1,1) \\ \{2,3\} & w=(2,2) \\ \{2,3\} & w=(2,3) \\ \{3\} & w=(3,2) \\ \{3\} & w=(3,3) \end{cases} (\mathcal{P}p_2)^\dagger \cdot \sigma_6(w) = \begin{cases} \{2\} & w=(1,2) \\ \{1\} & w=(2,1) \\ \{2\} & w=(1,1) \\ \{3\} & w=(2,2) \\ \{3\} & w=(2,3) \\ \{2\} & w=(3,2) \\ \{3\} & w=(3,3) \end{cases}$$

$$\sigma_2' = \sigma_5' = \sigma_6' = \sigma_3 = \sigma_4''$$

If we define $\sqsubseteq$ on simulations as

$$\sigma_1 \sqsubseteq \sigma_2 \iff \forall x_1, x_2 \in X, (\mathcal{P}p_i)^\dagger \cdot \sigma_1(x_1,x_2) \subseteq (\mathcal{P}p_i)^\dagger \cdot \sigma_2(x_1,x_2)$$

Lemma 2.15. *$(Hom(R, (\mathcal{P}R)^\dagger), \sqsubseteq)$ is a poset.*

Proof. Reflexivity and transitivity are obvious. We need to prove anti-symmetry.
TODO: Finish!

Then we have

$$\begin{array}{ccccccc} \sigma_6 & & & & \sigma_1 & & \\ & \rotatebox{45}{$\sqsubseteq$} & & \rotatebox{-45}{$\sqsupseteq$} & & \rotatebox{45}{$\sqsubseteq$} & \\ \sigma_5 & \sqsubseteq & \sigma_2 & \sqsubseteq & \sigma_3 & \sqsubseteq & \sigma_4 \end{array}$$

We recall that in the above diagram σ_3 is a bisimulation, and the rest are simulations.

We can also define $\sqcup$ and $\sqcap$ on morphisms as follows:

$$\forall x_1, x_2 \in X,$$
$$\sigma_1 \sqcup \sigma_2(x_1, x_2) = \sigma_1(x_1, x_2) \cup \sigma_2(x_1, x_2),$$
$$\sigma_1 \sqcap \sigma_2(x_1, x_2) = (\mathcal{P}p_1)^\dagger \cdot \sigma_1(x_1, x_2) \cap (\mathcal{P}p_1)^\dagger \cdot \sigma_2(x_1, x_2) \times (\mathcal{P}p_2)^\dagger \cdot \sigma_1(x_1, x_2) \cap (\mathcal{P}p_2)^\dagger \cdot \sigma_2(x_1, x_2).$$

{lem:proj-dist-set}

Lemma 2.16. *For relations R_1 and R_2 the following equation holds:*

$$(\mathcal{P}p_i)(R_1 \cup R_2) = (\mathcal{P}p_i)(R_1) \cup (\mathcal{P}p_i)(R_2)$$

Proof. We prove the lemma for the case that $i = 1$. The proof is the same for $i = 2$. Assuming $x_1 \in (\mathcal{P}p_1)^\dagger(R_1 \cup R_2)$ then exists x_2 that $(x_1, x_2) \in R_1 \cup R_2$, thus either $(x_1, x_2) \in R_1$ or $(x_1, x_2) \in R_2$, so we have $x_1 \in (\mathcal{P}p_1)^\dagger(R_1)$ or $x_1 \in (\mathcal{P}p_1)^\dagger(R_2)$, respectively. So, we have $x_1 \in (\mathcal{P}p_1)^\dagger(R_1) \cup (\mathcal{P}p_1)^\dagger(R_2)$.

Now, assuming that $x_1 \in (\mathcal{P}p_1)^\dagger(R_1) \cup (\mathcal{P}p_1)^\dagger(R_2)$ either $x_1 \in (\mathcal{P}p_1)^\dagger(R_1)$ or $x_1 \in (\mathcal{P}p_1)^\dagger(R_2)$. Without loss of generality, we can assume $x_1 \in (\mathcal{P}p_1)^\dagger(R_j)$, where $j \in \{1, 2\}$. Then there exists x_2 that $(x_1, x_2) \in R_j$, then we have $(x_1, x_2) \in R_1 \cup R_2$ that gives $x_1 \in (\mathcal{P}p_1)^\dagger(R_1 \cup R_2)$.

Lemma 2.17. *Assuming that σ_1 and σ_2 are simulation structures of type $R \to (\mathcal{P}R)^\dagger$, then $\sigma_1 \sqcup \sigma_2$ and $\sigma_1 \sqcap \sigma_2$ are also simulation structures of the same type.*

Proof. Since σ_1 and σ_2 are simulation structures, for every $(x_1, x_2) \in R$, for $i \in \{1, 2\}$ we have:

$$\alpha(x_1) \subseteq (\mathcal{P}p_1)^\dagger \cdot \sigma_i(x_1, x_2), \quad (6)$$
$$(\mathcal{P}p_2)^\dagger \cdot \sigma_i(x_1, x_2) \subseteq \alpha(x_2). \quad (7)$$

First, we prove the case for $\sqcup$. Since $\alpha(x_1) \subseteq (\mathcal{P}p_1)^\dagger \cdot \sigma_i(x_1, x_2)$ we have the following:

$$\alpha(x_1) \subseteq (\mathcal{P}p_1)^\dagger \cdot \sigma_1(x_1, x_2) \cup (\mathcal{P}p_1)^\dagger \cdot \sigma_2(x_1, x_2)$$

So, by Lemma 2.16 we have $\alpha(x_1) \subseteq (\mathcal{P}p_1)^\dagger(\sigma_1(x_1,x_2) \cup \sigma_2(x_1,x_2))$. Similarly, we have $(\mathcal{P}p_2)^\dagger \cdot \sigma_i(x_1,x_2) \subseteq \alpha(x_2)$ that gives the following:

$$(\mathcal{P}p_2)^\dagger \cdot \sigma_1(x_1,x_2) \cup (\mathcal{P}p_2)^\dagger \cdot \sigma_2(x_1,x_2) \subseteq \alpha(x_2)$$

So, by Lemma 2.16 we have $(\mathcal{P}p_2)^\dagger(\sigma_1(x_1,x_2) \cup \sigma_2(x_1,x_2)) \subseteq \alpha(x_2)$.

Now, we prove the case for $\sqcap$. For $\sqcap$ unlike $\sqcup$ we need to prove that $\sigma_1 \sqcap \sigma_2(x_1,x_2) \in (\mathcal{P}R)^\dagger$. To achieve this, we need to show that assuming π_1, π_2 are projections of $\sigma_1 \sqcap \sigma_2(x_1,x_2)$, then for $j \in \{1,2\}$ we have $\pi_j \cdot (\sigma_1 \sqcap \sigma_2)(x_1,x_2) \subseteq \mathcal{P}p_j(R)$. Since $(\mathcal{P}p_j)^\dagger \cdot \sigma_i(x_1,x_2) \subseteq \mathcal{P}p_j(R)$, we have $\pi_j \cdot (\sigma_1 \sqcap \sigma_2)(x_1,x_2) \subseteq \mathcal{P}p_j(R)$, so we have $\sigma_1 \sqcap \sigma_2(x_1,x_2) \in (\mathcal{P}R)^\dagger$, meaning that $\pi_j \cdot (\sigma_1 \sqcap \sigma_2)(x_1,x_2) = (\mathcal{P}p_j)^\dagger \cdot (\sigma_1 \sqcap \sigma_2)(x_1,x_2)$.[1]

For $j \in \{1,2\}$ we have

{eq:proj-meet}

$$(\mathcal{P}p_j)^\dagger \cdot (\sigma_1 \sqcap \sigma_2(x_1,x_2)) = (\mathcal{P}p_j)^\dagger \cdot \sigma_1(x_1,x_2) \cap (\mathcal{P}p_j)^\dagger \cdot \sigma_2(x_1,x_2). \quad (8)$$

Since $\alpha(x_1) \subseteq (\mathcal{P}p_1)^\dagger \cdot \sigma_i(x_1,x_2)$, we have

$$\alpha(x_1) \subseteq (\mathcal{P}p_1)^\dagger \cdot \sigma_1(x_1,x_2) \cap (\mathcal{P}p_1)^\dagger \cdot \sigma_2(x_1,x_2),$$

so by (8) we have $\alpha(x_1) \subseteq (\mathcal{P}p_1)^\dagger \cdot (\sigma_1 \sqcap \sigma_2(x_1,x_2))$. Similarly, since $(\mathcal{P}p_2)^\dagger \cdot \sigma_i(x_1,x_2) \subseteq \alpha(x_2)$, we have

$$(\mathcal{P}p_2)^\dagger \cdot \sigma_1(x_1,x_2) \cap (\mathcal{P}p_2)^\dagger \cdot \sigma_2(x_1,x_2) \subseteq \alpha(x_2),$$

so by (8) we have $(\mathcal{P}p_2)^\dagger \cdot (\sigma_1 \sqcap \sigma_2(x_1,x_2)) \subseteq \alpha(x_2)$. □

{lem:sim-opsim-inc}

Lemma 2.18. *Assuming that $\sigma\colon R \to (\mathcal{P}R)^\dagger$ is a simulation structure, and R is symmetric, then for all $(x_1,x_2) \in R$ we have:*

1. $(\mathcal{P}p_1)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma \cdot s(x_1,x_2) \subseteq (\mathcal{P}p_1)^\dagger \cdot \sigma(x_1,x_2)$
2. $(\mathcal{P}p_2)^\dagger \cdot \sigma(x_1,x_2) \subseteq (\mathcal{P}p_2)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma \cdot s(x_1,x_2)$

Proof. We prove the second clause. By (4) for every $(x_1,x_2) \in R$ we have

$$\alpha(x_1) \subseteq (\mathcal{P}p_1)^\dagger \cdot \sigma(x_1,x_2),$$
$$(\mathcal{P}p_2)^\dagger \cdot \sigma(x_1,x_2) \subseteq \alpha(x_2).$$

From $\alpha(x_1) \subseteq (\mathcal{P}p_1)^\dagger \cdot \sigma(x_1,x_2)$ since R is symmetric we get $\alpha(x_2) \subseteq (\mathcal{P}p_1)^\dagger \cdot \sigma(x_2,x_1)$, where

$$(\mathcal{P}p_1)^\dagger \cdot \sigma(x_2,x_1) = (\mathcal{P}p_2)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma \cdot s(x_1,x_2).$$

So, from $(\mathcal{P}p_2)^\dagger \cdot \sigma(x_1,x_2) \subseteq \alpha(x_2)$ we have $(\mathcal{P}p_2)^\dagger \cdot \sigma(x_1,x_2) \subseteq (\mathcal{P}p_2)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma \cdot s(x_1,x_2)$. Similarly, we can get the other inequation. □

[1] PP Note: The last part of the proof is necessary because the type of the codomain of the definition of $\sqcap$ is not $(\mathcal{P}R)^\dagger$, but it is $\mathcal{P}X \times \mathcal{P}X$. Perhaps the epi-mono factorization must be used to cope with this in the abstract case.

{lem:sim-bisim-inc}

Lemma 2.19. *Assuming that $\sigma\colon R \to (\mathcal{P}R)^\dagger$ is a simulation structure, and $\beta\colon R \to (\mathcal{P}R)^\dagger$ is a bisimulation structure,*

1. *if $\sigma \sqsubseteq \beta$ then we have:*

$$\alpha(x_1) = (\mathcal{P}p_1)^\dagger \cdot \sigma(x_1, x_2),$$

and if R is symmetric we have

$$(\mathcal{P}p_2)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma \cdot s(x_1, x_2) = \alpha(x_2).$$

2. *if $\beta \sqsubseteq \sigma$ then we have:*

$$(\mathcal{P}p_2)^\dagger \cdot \sigma(x_1, x_2) = \alpha(x_2)$$

and if R is symmetric we have

$$\alpha(x_1) = (\mathcal{P}p_1)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma \cdot s(x_1, x_2).$$

Proof. 1. Since σ is a simulation structure for an arbitrary $(x_1, x_2) \in R$ we have $\alpha(x_1) \subseteq (\mathcal{P}p_1)^\dagger \cdot \sigma(x_1, x_2)$. Since $\sigma \sqsubseteq \beta$ we have $(\mathcal{P}p_1) \cdot \sigma(x_1, x_2) \subseteq (\mathcal{P}p_1) \cdot \beta(x_1, x_2)$, while $(\mathcal{P}p_1) \cdot \beta(x_1, x_2) = \alpha(x_1)$ by definition of bisimulation. So we have $\alpha(x_1) = (\mathcal{P}p_1)^\dagger \cdot \sigma(x_1, x_2)$. Then because of the symmetry of R the second clause is easily achievable by using the equations in (5).

2. This clause can be proven similar to (1).

□

Proposition 2.20. *Assuming that $\sigma\colon R \to (\mathcal{P}R)^\dagger$ is a simulation structure, and $\beta\colon R \to (\mathcal{P}R)^\dagger$ is a bisimulation structure,*

1. *if $\sigma \sqsubseteq \beta$ then we have:*

$$\beta = \sigma \sqcup ((\mathcal{P}s)^\dagger \cdot \sigma \cdot s)$$

2. *if $\beta \sqsubseteq \sigma$ then we have:*

$$\beta = \sigma \sqcap ((\mathcal{P}s)^\dagger \cdot \sigma \cdot s)$$

Proof. 1. We need to prove that $\sigma \sqcup ((\mathcal{P}s)^\dagger \cdot \sigma \cdot s)$ is the bisimulation structure. By Lemma 2.18.(1), for every $(x_1, x_2) \in R$, we have $(\mathcal{P}p_1)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma \cdot s(x_1, x_2) \subseteq (\mathcal{P}p_1)^\dagger \cdot \sigma(x_1, x_2)$, and by Lemma 2.19.(1), we have $(\mathcal{P}p_1)^\dagger \cdot \sigma(x_1, x_2) = \alpha(x_1)$. So, we have $(\mathcal{P}p_1)^\dagger \cdot \sigma \sqcup (\mathcal{P}p_1)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma \cdot s(x_1, x_2) = \alpha(x_1)$, then by Lemma 2.16 we have $(\mathcal{P}p_1)^\dagger \cdot (\sigma \sqcup ((\mathcal{P}s)^\dagger \cdot \sigma \cdot s))(x_1, x_2) = \alpha(x_1)$.

Also, by Lemma 2.19.(1) we have $(\mathcal{P}p_2)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma(x_1, x_2) = \alpha(x_2)$. So, since we already have $(\mathcal{P}p_2)^\dagger \cdot \sigma(x_1, x_2) \subseteq \alpha(x_2)$ then by Lemma 2.16 we have $(\mathcal{P}p_2)^\dagger \cdot (\sigma \sqcup ((\mathcal{P}s)^\dagger \cdot \sigma \cdot s))(x_1, x_2) = \alpha(x_2)$.

2. We need to prove that $\sigma \sqcap ((\mathcal{P}s)^\dagger \cdot \sigma \cdot s)$ is the bisimulation structure. For $i \in \{1, 2\}$, for every $(x_1, x_2) \in R$, we have:

$(\mathcal{P}p_i)^\dagger \cdot (\sigma \sqcap ((\mathcal{P}s)^\dagger \cdot \sigma \cdot s))(x_1, x_2)$

$$= (\mathcal{P}p_i)^\dagger \cdot (((\mathcal{P}p_1)^\dagger \cdot \sigma(x_1,x_2) \cap (\mathcal{P}p_1)^\dagger \cdot ((\mathcal{P}s)^\dagger \cdot \sigma \cdot s)(x_1,x_2)) \times ((\mathcal{P}p_2)^\dagger \cdot \sigma(x_1,x_2) \cap (\mathcal{P}p_2)^\dagger \cdot ((\mathcal{P}s)^\dagger \cdot \sigma \cdot s)(x_1,x_2)))$$
$$= (\mathcal{P}p_i)^\dagger \cdot \sigma(x_1,x_2) \cap (\mathcal{P}p_i)^\dagger \cdot ((\mathcal{P}s)^\dagger \cdot \sigma \cdot s)(x_1,x_2)$$

By Lemma 2.18.(1), $(\mathcal{P}p_1)^\dagger \cdot ((\mathcal{P}s)^\dagger \cdot \sigma \cdot s)(x_1,x_2) \subseteq (\mathcal{P}p_1)^\dagger \cdot \sigma(x_1,x_2)$, and by Lemma 2.19.(2) we have $(\mathcal{P}p_1)^\dagger \cdot ((\mathcal{P}s)^\dagger \cdot \sigma \cdot s)(x_1,x_2) = \alpha(x_1)$, so we have $(\mathcal{P}p_1)^\dagger \cdot (\sigma \sqcap ((\mathcal{P}s)^\dagger \cdot \sigma \cdot s))(x_1,x_2) = \alpha(x_1)$.

Also, by Lemma 2.19.(2) we have $(\mathcal{P}p_2)^\dagger \cdot \sigma(x_1,x_2) = \alpha(x_2)$, so, since by Lemma 2.18.(2), we have $(\mathcal{P}p_2)^\dagger \cdot \sigma(x_1,x_2) \subseteq (\mathcal{P}p_2)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma \cdot s(x_1,x_2)$, so we have $(\mathcal{P}p_2)^\dagger \cdot (\sigma \sqcap ((\mathcal{P}s)^\dagger \cdot \sigma \cdot s)(x_1,x_2)) = \alpha(x_2)$. □

Corollary 2.21. *Assuming that R is a symmetric relation, and $S \neq \emptyset$ is the set of all simulation structures of the type $R \to (\mathcal{P}R)^\dagger$, then if the bisimulation morphism exists, it is equal with the following morphism:*

$$(\bigsqcup_{\sigma\in S} \sigma) \sqcap (\mathcal{P}s)^\dagger \cdot (\bigsqcup_{\sigma\in S} \sigma) \cdot s$$

{lem:alph-prod}

Lemma 2.22. *Assuming that R is a symmetric relation, and $S \neq \emptyset$ is the set of all simulation structures of the type $R \to (\mathcal{P}R)^\dagger$, then there there exists a simulation structure $\sigma \in S$ that for every (x_1,x_2), $(\mathcal{P}p_1)^\dagger \cdot \sigma(x_1,x_2) = \alpha(x_1)$.*

Proof. Since $S \neq \emptyset$ there exists $\delta \in S$. We define σ for every (x_1,x_2) as the following:

$$\sigma(x_1,x_2) = \alpha(x_1) \times (\mathcal{P}p_2)^\dagger \cdot \delta(x_1,x_2)$$

We have $\sigma(x_1,x_2) \in (\mathcal{P}R)^\dagger$, as $\alpha(x_1) \subseteq \mathcal{P}p_1(R)$ and $(\mathcal{P}p_2)^\dagger \cdot \delta(x_1,x_2) \subseteq \mathcal{P}p_2(R)$ are inherited from δ being a simulation structure. Also, it obviously is a simulation as $(\mathcal{P}p_1)^\dagger \cdot \sigma(x_1,x_2) = \alpha(x_1)$ and $(\mathcal{P}p_2)^\dagger \cdot \sigma(x_1,x_2) \subseteq \alpha(x_2)$ as $(\mathcal{P}p_2)^\dagger \cdot \delta(x_1,x_2) \subseteq \alpha(x_2)$.

Proposition 2.23. *Assuming that R is a symmetric relation, and $S \neq \emptyset$ is the set of all simulation structures of the type $R \to (\mathcal{P}R)^\dagger$, then the following morphism is the bisimulation structure:*

$$(\bigsqcap_{\sigma\in S} \sigma) \sqcup (\mathcal{P}s)^\dagger \cdot (\bigsqcap_{\sigma\in S} \sigma) \cdot s$$

Proof. For every $(x_1,x_2) \in R$ we have

$$(\mathcal{P}p_1)^\dagger \cdot ((\bigsqcap_{\sigma\in S} \sigma) \sqcup (\mathcal{P}s)^\dagger \cdot (\bigsqcap_{\sigma\in S} \sigma) \cdot s)(x_1,x_2) = (\mathcal{P}p_1)^\dagger \cdot (\bigsqcap_{\sigma\in S} \sigma)(x_1,x_2),$$

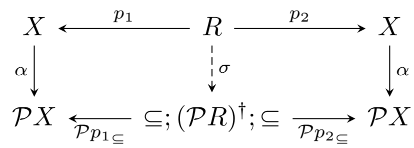
and

$$(\mathcal{P}p_2)^\dagger \cdot ((\bigsqcap_{\sigma\in S} \sigma) \sqcup (\mathcal{P}s)^\dagger \cdot (\bigsqcap_{\sigma\in S} \sigma) \cdot s)(x_1,x_2) = (\mathcal{P}p_2)^\dagger \cdot ((\mathcal{P}s)^\dagger \cdot (\bigsqcap_{\sigma\in S} \sigma) \cdot s)(x_1,x_2).$$

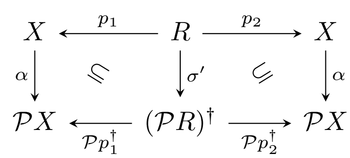
By Lemma 2.22 there exists a simulation $\delta \in S$ for which we have $(\mathcal{P}p_1)^\dagger \cdot \delta(x_1,x_2) = \alpha(x_1)$. So, $(\mathcal{P}p_1)^\dagger \cdot (\bigsqcap_{\sigma\in S} \sigma)(x_1,x_2) = \alpha(x_1)$. Then by the equations in (5) we also get $(\mathcal{P}p_2)^\dagger \cdot ((\mathcal{P}s)^\dagger \cdot (\bigsqcap_{\sigma\in S} \sigma) \cdot s)(x_1,x_2) = \alpha(x_2)$. □

3 Relators

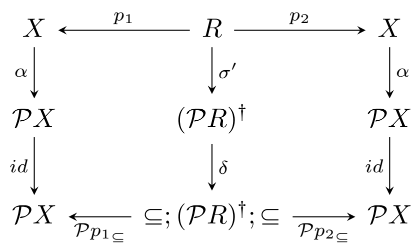
We start the discussion with answering the question that why there can be multiple simulation structures based on Definition 2.8. At the moment we have limited the discussion to the category of sets and we are talking about the powerset functor. We know that σ is unique in the following diagram:



It is defined as $\sigma(x_1, x_2) = (\alpha(x_1), \alpha(x_2))$. But σ' in the following diagram is not unique:



Because assuming we have σ, for every given σ' we can define a $\delta\colon (\mathcal{P}R)^\dagger \to \subseteq; (\mathcal{P}R)^\dagger; \subseteq$ that $\sigma = \delta \cdot \sigma'$, i.e., the following diagram commutes:



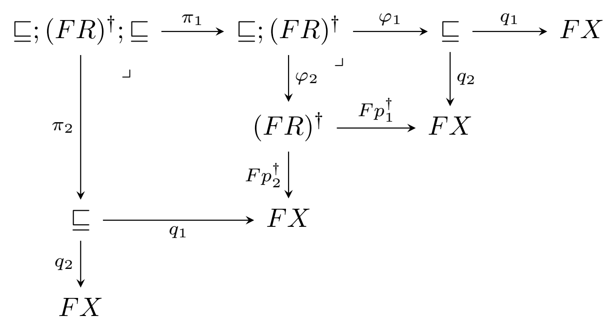
To define δ, we define $c\colon (\mathcal{P}R^\dagger) \to ((\mathcal{P}R^\dagger) \times R) + (\mathcal{P}R)^\dagger$ and $u\colon ((\mathcal{P}R^\dagger) \times R) + (\mathcal{P}R)^\dagger \to \subseteq; (\mathcal{P}R)^\dagger; \subseteq$ and then we define $\delta = u \cdot c$. Here are the definitions for c and u:

$$c(w) = \begin{cases} \mathsf{inl}(w, (x_1, x_2)) & \exists x_1, x_2, \sigma'(x_1, x_2) = w \\ \mathsf{inr}\, w & \mathsf{o.w} \end{cases}$$

$$u(\mathsf{inl}\, w, (x_1, x_2)) = (\alpha(x_1), \alpha(x_2))$$

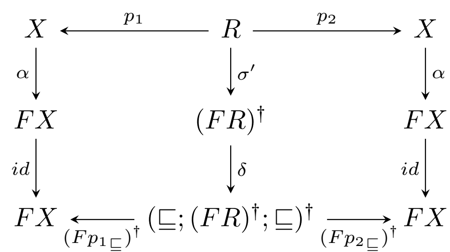
$$u(\mathsf{inr}\, w) = w$$

Now, we want to prove a more abstract version of this statement. First, we need to spell out what $\sqsubseteq; (FR)^\dagger; \sqsubseteq$ is. We define relation compositions with pullbacks, so we have the following diagram:

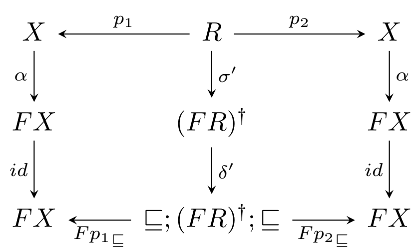


Additionally, we make the abbreviations that $Fp_{1_\sqsubseteq} = q_1 \cdot \varphi_1 \cdot \pi_1$ and $Fp_{2_\sqsubseteq} = q_2 \cdot \pi_2$.

Proposition 3.1. *Assuming we have a morphism $\sigma'\colon R \to (FR)^\dagger$ then exists $\delta\colon (FR)^\dagger \to (\sqsubseteq;(FR)^\dagger;\sqsubseteq)^\dagger$ such that $\sigma = \delta \cdot \sigma'$ that commutes in the following diagram:*



Proof. Ultimately, we need to define $\delta'\colon (FR) \to \sqsubseteq;(FR)^\dagger;\sqsubseteq$, where $\delta = e_{\sqsubseteq;(FR)^\dagger;\sqsubseteq} \cdot \delta'$, and $e_{\sqsubseteq;(FR)^\dagger;\sqsubseteq}\colon \sqsubseteq;(FR)^\dagger;\sqsubseteq \to (\sqsubseteq;(FR)^\dagger;\sqsubseteq)^\dagger$ is the epimorphism in the epi-mono factorization of $\langle Fp_{1_\sqsubseteq}, Fp_{2_\sqsubseteq}\rangle$ because by Lemma 2.7 it suffices to show that the following diagram commutes:



So, we need to define δ'.